Homework #1 (40 + 1 pts)
Solutions

1. *(1.1) Our First Constellation.*

   (a) (**2 pts**) The following two basis functions are orthonormal,

$$\int_0^1 \phi_1(t)\phi_2(t)dt = \int_0^1 2\sin(2\pi t)\cos(2\pi t)dt = \int_0^1 \sin(4\pi t)dt = 0$$
$$\int_0^1 \phi_1^2(t)dt = \int_0^1 2\cos^2(2\pi t)dt = \int_0^1 [1+\cos(4\pi t)]dt = 1$$
$$\int_0^1 \phi_2^2(t)dt = \int_0^1 2\sin^2(2\pi t)dt = \int_0^1 [1-\cos(4\pi t)]dt = 1$$

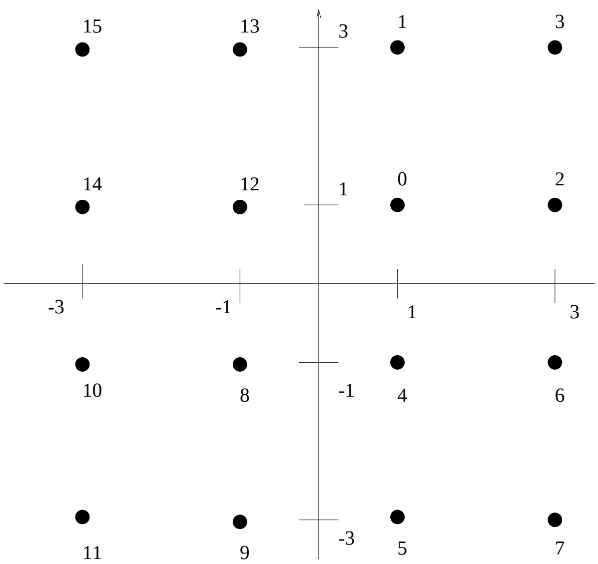

Figure 1: Signal Constellation.

   (b) (**2 pts**)The signal constellation is shown on figure 1.

   (c) i. (**2 pts**) For the case where all signals are equally likely, the average energy $\mathcal{E}_{\boldsymbol{x}}$ is given by,

$$\mathcal{E}_{\boldsymbol{x}} = \frac{1}{4}(2+18+10+10) = 10$$

And the average energy per dimension shall be,

$$\overline{\mathcal{E}}_{\boldsymbol{x}} = 10/2 = 5$$

ii. (**2 pts**) For the case where

$$p(x_0) = p(x_4) = p(x_8) = p(x_{12}) = \frac{1}{8}$$

and

$$p(x_i) = \frac{1}{24} \quad i = 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15$$

The average energy is simply,

$$\begin{aligned}
\mathcal{E}_{\boldsymbol{x}} &= \frac{4}{8}(2) + \frac{4}{24}(10 + 10 + 18), \\
&= \frac{22}{3}, \\
\overline{\mathcal{E}}_{\boldsymbol{x}} &= \frac{11}{3}.
\end{aligned}$$

(d) (**2 pts**) Note that $(\phi_1(t), \phi_2(t), \phi_3(t))$ forms an orthonormal basis. The signal constellation $\mathcal{Y}$ lives in a dimension one bigger than that of $\mathcal{X}$, and is a translation of $\mathcal{X}$. The corresponding energy is then,

$$\begin{aligned}
\mathcal{E}_{\boldsymbol{y}} &= \sum_{i=0}^{15} (||\boldsymbol{x}_i||^2 + 16) p_X(i) \\
&= \mathcal{E}_{\boldsymbol{x}} + 16 = 26
\end{aligned}$$

2. *(1.2) Inner Products.*

(a) (**4 pts**) By using the following trigonometric identity,

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

the signals $x_0(t), x_1(t), x_2(t)$ can be written as,

$$\begin{aligned}
x_0(t) &= \sqrt{2}[\phi_1(t)\cos(\frac{\pi}{6}) - \phi_2(t)\sin(\frac{\pi}{6})] \\
x_1(t) &= \sqrt{2}[\phi_1(t)\cos(\frac{5\pi}{6}) - \phi_2(t)\sin(\frac{5\pi}{6})] \\
x_2(t) &= \sqrt{2}[\phi_1(t)\cos(\frac{\pi}{2}) + \phi_2(t)\sin(\frac{\pi}{2})]
\end{aligned}$$

where

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}\left(\cos\left(\frac{2\pi t}{T}\right)\right) & \text{if } t \epsilon [0, T] \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}\left(\sin\left(\frac{2\pi t}{T}\right)\right) & \text{if } t \epsilon [0, T] \\ 0 & \text{otherwise} \end{cases}$$

are indeed an orthonormal basis,

$$\int_0^T \phi_1(t)\phi_2(t)dt = \int_0^T \frac{2}{T}\sin(\frac{2\pi t}{T})\cos(\frac{2\pi t}{T})dt = \int_0^T \sin(\frac{4\pi t}{T})dt = 0$$
$$\int_0^T \phi_1^2(t)dt = \int_0^T \frac{2}{T}\cos^2(\frac{2\pi t}{T})dt = \int_0^T \frac{1}{T}[1+\cos(\frac{4\pi t}{T})]dt = 1$$
$$\int_0^T \phi_2^2(t)dt = \int_0^T \frac{2}{T}\sin^2(\frac{2\pi t}{T})dt = \int_0^T \frac{1}{T}[1-\cos(\frac{4\pi t}{T})]dt = 1$$

(b) (**3 pts**) From part (a) it is easily seen that the data symbols are,

$$\begin{aligned} x_0 &= \begin{bmatrix} \sqrt{\frac{3}{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix} \\ x_1 &= \begin{bmatrix} -\sqrt{\frac{3}{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix} \\ x_2 &= \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix} \end{aligned}$$

(c) (**3 pts**) From the conservation of inner product,

$$\langle x(t), y(t)\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$$

we obtain,

i. $\langle x_0(t), x_0(t)\rangle = \frac{3}{2} + \frac{1}{2} = 2.$
ii. $\langle x_0(t), x_1(t)\rangle = -\frac{3}{2} + \frac{1}{2} = -1$
iii. $\langle x_0(t), x_2(t)\rangle = -1$

3. *(1.3) Multiple sets of basis functions.*

(a) (**2 pts**) The functions u(t) and v(t) are given by,

$$u(t) = \begin{cases} \frac{2}{3} & \text{if } t\epsilon[0, 2.25] \\ 0 & \text{if } t\epsilon[2.25, 6.75] \\ \frac{2}{3} & \text{if } t\epsilon[6.75, 9] \end{cases} \quad v(t) = \begin{cases} 1 & \text{if } t\epsilon[0, 2.25] \\ -\frac{1}{3} & \text{if } t\epsilon[2.25, 6.75] \\ 1 & \text{if } t\epsilon[6.75, 9] \end{cases}$$

See figure 2.

(b) (**3 pts**) The new basis functions are,

$$\phi_1(t) = \begin{cases} \frac{\sqrt{2}}{3} & \text{if } t\epsilon[0, 2.25] \\ 0 & \text{if } t\epsilon[2.25, 6.75] \\ \frac{\sqrt{2}}{3} & \text{if } t\epsilon[6.75, 9] \end{cases} \quad \phi_2(t) = \begin{cases} 0 & \text{if } t\epsilon[0, 2.25] \\ \frac{\sqrt{2}}{3} & \text{if } t\epsilon[2.25, 6.75] \\ 0 & \text{if } t\epsilon[6.75, 9] \end{cases}$$

See figure 3.
And obviously, we have

$$v = \begin{bmatrix} \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} \end{bmatrix}$$

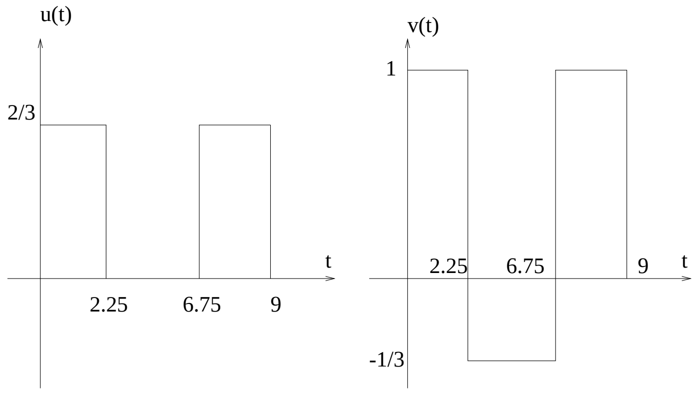

Figure 2: $u(t)$ and $v(t)$

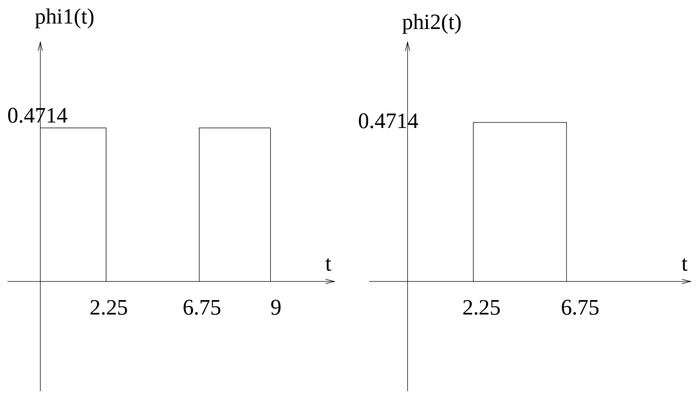

Figure 3: New Basis function

4. *(1.4) Minimal orthonormalization with MATLAB.*

   (a) (**2 pts**) MATLAB is used to generate the new set of basis functions,

```
A =
     1     2     3     4     5     6     7     8
     1     3     5     7     9    11    13    15
     2     4     6     8    10    12    14    16
     0     1     0     1     0     1     0     1
     0     2     0     2     0     2     0     2
     2     4     2     4     2     4     2     4
Q=orth(A)
Q =

    0.3394    0.0323    0.1124
    0.6190    0.2214   -0.3152
    0.6788    0.0647    0.2249
    0.0338   -0.2450   -0.3479
    0.0677   -0.4900   -0.6959
    0.1873   -0.8035    0.4563
```

Note that the Q matrix is not unique. MATLAB normally will give the above answer, except may be for a sign change of some of the basis vectors.

(b) (**2 pts**)The number of basis functions needed is 3. In the time domain, the new basis functions $\hat{\phi}_i(t)$, i=1,2,3, are given by,

$$\left[\begin{array}{ccc} \hat{\phi}_1(t) & \hat{\phi}_2(t) & \hat{\phi}_3(t) \end{array} \right] = \phi Q$$

where $\phi = [\phi_1(t)\ \phi_2(t)\ \phi_3(t)\ \phi_4(t)\ \phi_5(t)\ \phi_6(t)\ ]$.

(c) (**1 pt**) The new matrix $\hat{A}$ which gives the data symbol representation for the original modulated waveforms using the smaller set of basis functions found in (b) is given by,

```
A_hat=Q'*A

A_hat =
  Columns 1 through 8
  2.6907    6.1696    8.5609   12.0398   14.4311   17.9100   20.3013   23.7802
 -1.2239   -3.4513   -0.0148   -2.2422    1.1943   -1.0331    2.4034    0.1760
  1.1235    0.1561    0.8430   -0.1244    0.5625   -0.4049    0.2820   -0.6853
```

This problem can also be done using the **qr** function in MATLAB, which gives us the answer to all the parts at once.

5. *(1.5) Decision rules for binary channels.*

(a) (**5 pts**) **The Binary Symmetric Channel (BSC)**.
For equally likely inputs the MAP and ML decision rules are identical. In each case we wish to maximize $p_{\boldsymbol{y}|\boldsymbol{x}}(y|x_i)$ over the possible choices for x_i. The decision rules are shown below,

$$p < \frac{1}{2} \Rightarrow \hat{X} = Y$$
$$p > \frac{1}{2} \Rightarrow \hat{X} = 1 - Y$$

(b) (**5 pts**) **The Binary Erasure Channel (BEC)**.
Again, since we have equiprobable signals, the MAP and ML decision rules are the same. The decision rules are as follows,

$$p_1 < p_2 < \frac{1}{2} \Rightarrow \hat{X} = \begin{cases} Y & \text{if Y=0,1.} \\ 1 & \text{if Y=2.} \end{cases}$$
$$p_2 < p_1 < \frac{1}{2} \Rightarrow \hat{X} = \begin{cases} Y & \text{if Y=0,1.} \\ 0 & \text{if Y=2.} \end{cases}$$