Homework #2 (67 + 1 pts)
Solutions

1. *(1.7) Irrelevancy/Decision Regions.* (From Wozencraft and Jacobs)

   (a) i. **(1 pt)** The following relations can be written,

   $$\begin{aligned} y_1 &= x + n_1 \\ y_2 &= x + n_2 \\ y_3 &= x + n_1 + n_2 \end{aligned}$$

   Using the above equations, $y_3 = y_1 + n_2$, and,

   $$p_{\boldsymbol{y}_3|\boldsymbol{y}_1,\boldsymbol{x}}(y_3|y_1,x) = p_{\boldsymbol{y}_3|\boldsymbol{y}_1}(y_3|y_1) = p_{\mathbf{n_2}}(y_3 + y_1)$$

   Therefore, given y_1 only, y_3 is irrelevant.

   ii. **(1 pt)** In this case, we have,

   $$\begin{aligned} p_{\boldsymbol{y}_3|\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{x}}(y_3|y_1,y_2,x) &= \delta(y_3 + y_2 + y_1 + x), \\ p_{\boldsymbol{y}_3|\boldsymbol{y}_1,\boldsymbol{y}_2}(y_3|y_1,y_2) &= p_{\boldsymbol{x}}(y_1 + y_2 + y_3) \neq \delta(y_3 + y_2 + y_1 + x). \end{aligned}$$

   where $\delta(x)$ is the Dirac delta function.
   Therefore y_3 is relevant. This fact is easily seen by noticing that $x = y_1 + y_2 + y_3$. If y_1, y_2, y_3 are known, the transmitted message x can be determined with a zero probability of error.

   (b) **(1 pt)** Again, the following conditional probabilities are computed,

   $$\begin{aligned} p_{\boldsymbol{y}_2|\boldsymbol{y}_1,\boldsymbol{x}}(y_2|y_1,x) &= p_{\mathbf{n_2}}(y_2 - x) \\ &= \frac{1}{2}e^{-|y_2 - x|} \\ p_{\boldsymbol{y}_2|\boldsymbol{y}_1}(y_2|y_1) &= p_{\boldsymbol{y}_2|\boldsymbol{y}_1,\boldsymbol{x}}(y_2|y_1,1)p_{\boldsymbol{x}}(1) + p_{\boldsymbol{y}_2|\boldsymbol{y}_1,\boldsymbol{x}}(y_2|y_1,-1)p_{\boldsymbol{x}}(-1) \\ &= \frac{1}{4}[e^{-|y_2-1|} + e^{-|y_2+1|}] \end{aligned}$$

   Obviously, $p_{\boldsymbol{y}_2|\boldsymbol{y}_1,\boldsymbol{x}} \neq p_{\boldsymbol{y}_2|\boldsymbol{y}_1}$ and therefore y_2 is relevant, given y_1.

   (c) **(3 pts)** In this part of the problem, the MAP rule will be applied, and then specialized for equiprobable input signals. The decision region D_1 is obtained by applying the MAP rule,

   $$p_{\boldsymbol{y}_2,\boldsymbol{y}_1|\boldsymbol{x}}(y_2,y_1|1)p_{\boldsymbol{x}}(1) > p_{\boldsymbol{y}_2,\boldsymbol{y}_1|\boldsymbol{x}}(y_2,y_1|-1)p_{\boldsymbol{x}}(-1)$$

Since n_1 and n_2 are independent random variables, then,

$$\begin{aligned}\frac{p}{4}e^{-|y_1-1|-|y_2-1|} &> \frac{1-p}{4}e^{-|y_1+1|-|y_2+1|},\\ \Leftrightarrow K(y_1,y_2) &= |y_1-1|+|y_2-1|-|y_1+1|-|y_2+1| < \log\frac{p}{1-p}.\end{aligned}$$

where $p_{\boldsymbol{x}}(1)=p$.
The following cases are taken into consideration,

i. $y_1>1,\ y_2>1 \Rightarrow K(y_1,y_2)=-4$
ii. $y_1<-1,\ y_2>1 \Rightarrow K(y_1,y_2)=0$
iii. $y_1>1,\ y_2<-1 \Rightarrow K(y_1,y_2)=0$
iv. $y_1<-1,\ y_2<-1 \Rightarrow K(y_1,y_2)=4$
v. $y_1>1,\ -1<y_2<1 \Rightarrow K(y_1,y_2)=-2(y_2+1)$
vi. $y_1<-1,\ -1<y_2<1 \Rightarrow K(y_1,y_2)=-2(y_2-1)$
vii. $-1<y_1<1,\ y_2>1 \Rightarrow K(y_1,y_2)=-2(y_1+1)$
viii. $-1<y_1<1,\ y_2<-1 \Rightarrow K(y_1,y_2)=-2(y_1-1)$
ix. $-1<y_1<1,\ -1<y_2<1 \Rightarrow K(y_1,y_2)=-2(y_1+y_2)$

For $p=\frac{1}{2}$ the boundaries of the decision regions are given by the equation $K(y_1,y_2)=0$, leading to the desired result.

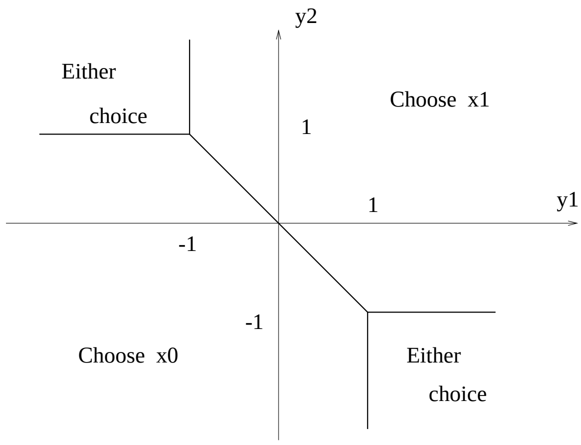

Figure 1: Optimum decision region for equiprobable messages.

Note: If $K(y_1,y_2)=0$ then either symbol x_i might be chosen.

(d) **(4 pts)** The receiver is optimum since its decision regions coincide with the ones of the optimum receiver derived earlier. The probability of error is given by,

$$P_e = Pr\{y_1+y_2>0|x=-1\}p_x(-1)+Pr\{y_1+y_2<0|x=1\}p_x(1)$$

Because of the symmetry of the pdf,
$Pr\{y_1+y_2>0|x=-1\}=Pr\{y_1+y_2<0|x=1\}=Pr\{n_1+n_2>2\}$
Let us define the random variable n as $n=n_1+n_2$. The pdf of n is the convolution of $f_{n_1}(n)$ and $f_{n_2}(n)$,

$$f_n(n)=f_{n_1}(n)*f_{n_2}(n)$$

Taking the Laplace transform on both sides, yields, after simplification,

$$F_n(s) = \mathcal{L}[f_{n_1}(n)]^2 = \frac{1}{(1-s^2)^2}$$

Using partial fraction expansion,

$$F_n(s) = \frac{1}{4(1-s)^2} + \frac{1}{4(1+s)^2} + \frac{1}{2(1-s^2)}$$

Taking the inverse Laplace transform gives,

$$f_n(t) = \frac{1}{4}(|t|+1)e^{-|t|}$$

Finally,

$$Pr\{n_1 + n_2 > 2\} = \int_2^\infty \frac{1}{4}(t+1)e^{-t}dt = e^{-2}$$

And the probability of error is, $P_e = e^{-2}$.

(e) **(2 pts)** From part (c),the decision regions for $x = 1$ are given by the inequality $K(y_1, y_2) < \log \frac{p}{1-p}$. Since $p > \frac{1}{2}$, $\log \frac{p}{1-p} > 0$. Now for each case, we try to solve the inequality,

i. $y_1 > 1,\ y_2 > 1 \Rightarrow K(y_1, y_2) = -2 < \log \frac{p}{1-p}$.
Therefore, in this region, choose $x = 1$.

ii. $y_1 < -1,\ y_2 > 1 \Rightarrow K(y_1, y_2) = 0 < \log \frac{p}{1-p}$.
In this region, choose $x = 1$.

iii. $y_1 > 1,\ y_2 < -1 \Rightarrow K(y_1, y_2) = 0 < \log \frac{p}{1-p}$.
In this region, choose $x = 1$.

iv. $y_1 < -1,\ y_2 < -1 \Rightarrow K(y_1, y_2) = 4$
In this case, we choose $x = 0$ for $\frac{1}{2} < p < \frac{1}{1+e^{-4}}$ and $x = 1$ for $\frac{1}{1+e^{-4}} < p < 1$.

v. $y_1 > 1,\ -1 < y_2 < 1 \Rightarrow K(y_1, y_2) = -2(y_2 + 1) < 0 < \log \frac{p}{1-p}$.
Therefore choose $x = 1$

vi. $y_1 < -1,\ -1 < y_2 < 1 \Rightarrow K(y_1, y_2) = -2(y_2 - 1) < \log \frac{p}{1-p}$.
Therefore, choose $x = 1$ if $y_2 > 1 - \frac{1}{2}\log \frac{p}{1-p}$.

vii. $-1 < y_1 < 1,\ y_2 > 1 \Rightarrow K(y_1, y_2) = -2(y_1 + 1) < 0 < \log \frac{p}{1-p}$.
Always choose $x = 1$.

viii. $-1 < y_1 < 1,\ y_2 < -1 \Rightarrow K(y_1, y_2) = -2(y_1 - 1) < \log \frac{p}{1-p}$.
Therefore choose $x = 1$ if $y_1 > 1 - \frac{1}{2}\log \frac{p}{1-p}$.

ix. $-1 < y_1 < 1,\ -1 < y_2 < 1 \Rightarrow K(y_1, y_2) = -2(y_1 + y_2) < \log \frac{p}{1-p}$
Therefore choose $x = 1$ if $y_1 + y_2 > -\frac{1}{2}\log \frac{p}{1-p}$

If we draw such regions, we get the figure shown below.

You should note that for $.982 < p < 1$ it turns out that the optimum receiver always chooses $x = 1$. And therefore, the outputs of the channel y_1 and y_2 are irrelevant.

2. *(1.9) Receiver Noise with MATLAB.*

We need to compute the $Q = orth(A)$ matrix. From matlab, we have,

$$Q = \begin{bmatrix} 0.3803 & 0.2260 & -0.0120 \\ 0.7606 & 0.4520 & -0.0239 \\ 0.0245 & -0.2395 & -0.0876 \\ 0.0464 & 0.1157 & 0.3710 \\ 0.2129 & -0.3713 & 0.8501 \\ 0.4783 & -0.7321 & -0.3624 \end{bmatrix}$$

(a) **(2 pts)** At the output of the basis detector, we have,

$$\boldsymbol{y} = (\boldsymbol{x} + \mathbf{n})$$

The signal power is given by,

$$E[\|\boldsymbol{x}\|^2] = \frac{\sum_{i=0}^{7} \|\boldsymbol{x}_i\|^2}{8} = \frac{trace(A^T A)}{8} = 181$$

The noise power can be found in any of the possible ways. All of them are given equal credit as the definition was not clearly specified.

i. $\sigma^2 = E[\|\mathbf{n}\|^2] = 6$. Hence, $SNR = 30.17 = 14.8dB$

ii. $\sigma^2 = E[\|\mathbf{n} - E[\mathbf{n}]\|^2] = 3$. Hence, $SNR = 60.33 = 17.8dB$

iii. In class, when we use AWGN noise, σ^2 is the variance of noise vector per dimension. This changes (i) to $\sigma^2 = \frac{1}{6}E[\|\mathbf{n}\|^2] = 1$. Hence, $SNR = 181 = 22.6dB$

iv. The argument in (iii) changes (ii) to $\sigma^2 = \frac{1}{6}E[\|\mathbf{n}\|^2] = 0.5$. Hence, $SNR = 362 = 25.6dB$

(b) **(4pts)** The basis detector is now using $\hat{\phi}(t) = \phi(t)Q$, where Q=orth(A) as in problem 1.4. In this case, the received vectors y at the output of the basis detector are,

$$\begin{aligned} \boldsymbol{y} &= Q^T(\boldsymbol{x} + \mathbf{n}) \\ &= (\hat{\boldsymbol{x}} + \hat{\mathbf{n}}) \end{aligned}$$

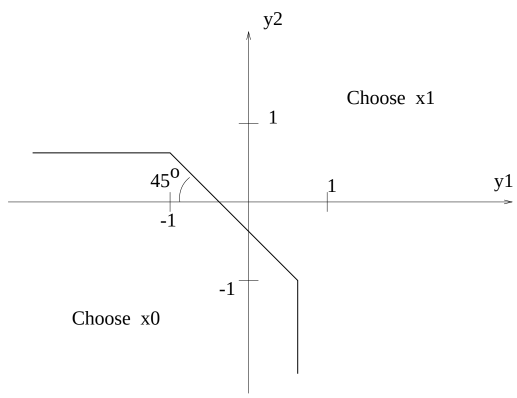

Figure 2: Optimum decision region for $0 \le p < 0.982$.

The signal power is unchanged as columns of Q form a basis for the symbol vectors. But, noise power may change as it may not lie completely in the space spanned by the columns of Q. As in (a) we give credit to any of the following answers. But, your answers in both parts have to be consistent.

i. $\sigma^2 = E[\|\hat{\mathbf{n}}\|^2] = 2.77$. Hence, $SNR = 65.3 = 18.1dB$
ii. $\sigma^2 = E[\|\hat{\mathbf{n}} - E[\hat{\mathbf{n}}]\|^2] = 1.68$. Hence, $SNR = 107.8 = 20.3dB$
iii. $\sigma^2 = \frac{1}{3}E[\|\hat{\mathbf{n}}\|^2] = 0.92$. Hence, $SNR = 195.9 = 22.9dB$
iv. $\sigma^2 = \frac{1}{3}E[\|\hat{\mathbf{n}}\|^2] = 0.56$. Hence, $SNR = 323.3 = 25.1dB$

(c) **(2 pts)** The error performance of these two detectors for AWGN will be identical because the extra noise admitted by the first detector is orthogonal to the space spanned by the signal set. For AWGN, this orthogonal noise is irrelevant. If the noise were correlated, you might even desire the first detector and use the orthogonal noise to infer something about the noise in the space spanned by the signal set.

(d) **(2 pts)** Using the Cauchy-Schwartz inequality, we choose $\mathbf{h} = \mathbf{x_1}$. And the maximum value of $\mathbf{h} \cdot \mathbf{x_1}$ is 66.

3. *(1.10) Tilt.*

(a) **(1 pt)** P_e is invariant under rotation or translation of signal constellation. Therefore, P_e does not depend on L or θ.

(b) **(2 pts)** First, we get the average number of nearest neighbors:

$$\begin{aligned} N_e &= \sum_{i=0}^{M-1} N_i p_{\boldsymbol{x}}(i) \\ &= \frac{4 + 4 \cdot 2 + 4 \cdot 3}{9} \\ &= \frac{8}{3}. \\ P_e &\leq \frac{8}{3} Q\left(\frac{d}{2\sigma}\right) \\ &= \frac{8}{3} Q\left(\frac{2}{2\sqrt{0.1}}\right) \\ &= 2.09 \times 10^{-3}. \end{aligned}$$

(c) **(5 pts)** By part (a), the constellation shown in the figure gives the same P_e:

$$\begin{aligned} P_{c|i=5} &= \left(1 - 2Q\left(\frac{d}{2\sigma}\right)\right)^2 \\ P_{c|i=2,4,6,8} &= \left(1 - 2Q\left(\frac{d}{2\sigma}\right)\right)\left(1 - Q\left(\frac{d}{2\sigma}\right)\right) \\ P_{c|i=1,3,5,9} &= \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)^2. \end{aligned}$$

Hence,

$$P_c = \frac{1}{9}\left[\left(1 - 4Q + 4Q^2\right) + 4\left(1 - 3Q + 2Q^2\right) + 4\left(1 - 2Q + Q^2\right)\right],$$

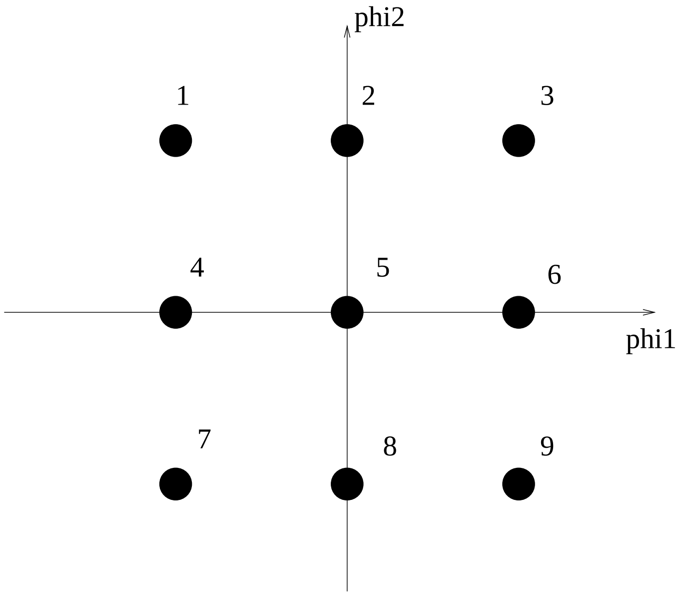

Figure 3: Equivalent Signal Constellation

$$\begin{aligned} P_e &= 1 - P_c \\ &= \frac{8}{3}Q - \frac{16}{9}Q^2, \end{aligned}$$

where $Q = Q\left(\frac{d}{2\sigma}\right) = Q(3.162) = 7.827 \times 10^{-4}$. So,

$$\begin{aligned} P_e &= 2.09 \times 10^{-3} - 1.09 \times 10^{-6} \\ &= 2.09 \times 10^{-3}. \end{aligned}$$

In this case, NNUB was off by $\frac{16}{9}Q^2 = 1.09 \times 10^{-6}$, a small quantity compared to P_e.

(d) **(2 pts)** To get a constellation with minimum energy, we subtract from the constellation its mean. So, a possible choice is the constellation of part c). The energy of the original constellation would change with θ, whereas the energy of the minimum-energy constellation would be independent of θ.

4. *(1.13) Rotation with correlated noise.*

(a) **(2 pts)** The noise n_L along the line connecting the two constellation points is simply, $n_L = n_1$. The mean and mean square values are then,

$$E[n_1] = 0 \text{ and } E[n_1^2] = 0.1$$

(b) **(2 pts)** Assuming that the detector was designed for uncorrelated noise (so we are using a ML detector), the probability of error is given by,

$$P_e = Q(\frac{d_{min}}{2\sigma}) = Q(\sqrt{10}) = 7.83 \times 10^{-4} \simeq 10^{-3},$$

where $d_{min} = 2$ and $\sigma^2 = 0.1$.

(c) **(2 pts)** For general angle θ, we need to project the noise components n_1 and n_2 on the line L connecting the two points x_1 and x_2. In order to do so, we need to find the angle of such a line with the horizontal and vertical axis.

From the figure, $\alpha = \frac{\pi}{4} - \theta$. Therefore, the noise along line L is given by,

$$n_L = n_1 \cos(\frac{\pi}{4} - \theta) - n_2 \sin(\frac{\pi}{4} - \theta)$$

And the corresponding mean and mean square values are,

$$\begin{aligned} E[n_L] &= 0 \\ E[n_L^2] &= E[n_1^2 \cos^2 \alpha + n_2^2 \sin^2 \alpha - n_1 n_2 \sin(2\alpha)] \\ &= E[n_1^2] - E[n_1 n_2] \cos 2\theta \\ &= 0.1 - 0.05 \cos 2\theta \end{aligned}$$

The mean square value of n_L is minimized when $\cos 2\theta = 1$, i.e. when $\theta = k\pi$. In this case, the probability of error is,

$$P_e = Q(\frac{1}{\sqrt{.05}}) = Q(\sqrt{20}) \simeq 3.87 \times 10^{-6}.$$

(d) **(1 pt)** The detector designed for AWGN makes its decision entirely based on the projection of the received signal onto the line L connecting the two signals points x_1 and x_2. Therefore, the detector can be improved if the noise orthogonal to line L is relevant. Let us denote this noise by n_o. In most of the cases, the orthogonal noise n_o will be correlated with the noise n_L, and will be useful to the optimum receiver. Hence, in general, our detector could be improved. Let us see for which values of θ the orthogonal noise n_o will be uncorrelated with n_L,

$$\begin{aligned} E[n_L n_o] &= E[n_L(n_1 \sin(\frac{\pi}{4} - \theta) + n_2 \cos(\frac{\pi}{4} - \theta)] \\ &= 0.05 \sin 2\theta \end{aligned}$$

The noises are uncorrelated if and only if $\theta = \frac{k\pi}{2}$. Thus, for the constellation in (c), the detector is optimal.

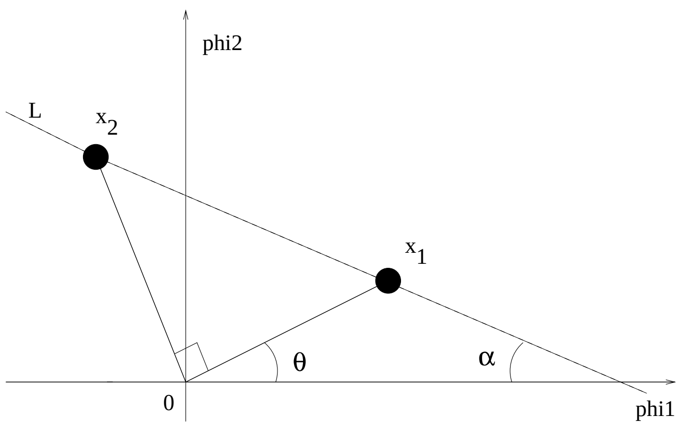

Figure 4: Constellation for general θ

5. *(1.16) Equivalency of rectangular-lattice constellations.*

(a) The $\overline{P}_e$ formula for PAM, QAM and TAM is identical,

$$\overline{P}_e \;\le\; 2 \cdot Q\left(\sqrt{\frac{3 \cdot SNR}{2^{2\bar{b}} - 1}}\right) \;<\; 10^{-6}$$

Therefore,

$$\begin{aligned} \frac{3 \cdot SNR}{2^{2\bar{b}} - 1} &= 13.8 \text{ dB} \\ 2^{2\bar{b}} - 1 &= 4.77 + 22 - 13.8 \;=\; 12.97 \text{ dB} \;\Rightarrow\; \bar{b} = 2.19 \end{aligned}$$

$\bar{b}$ must be an integer, and so the highest data rate corresponds to $\bar{b} = 2$ in all cases. Thus,

i. **(0.5 pts)** PAM : $R = \bar{b} \cdot N/T = 2 * 1 * 8k = 16$ kbps

ii. **(0.5 pts)** QAM : $R = 2 * 2 * 8k = 32$ kbps

iii. **(0.5 pts)** TAM : $R = 2 * 3 * 8k = 48$ kbps

(b) **(1.5 pts)** In all three cases,

$$\frac{3 \cdot SNR}{2^{2\bar{b}} - 1} \;=\; 4.77 + 22 - 11.76 = 15.01 \text{ dB}$$

$$\overline{P}_e \;\le\; 2 \cdot Q(15.01dB) \;=\; 1.8 \cdot 10^{-8}$$

(c) **(2 pts)** (*Remark :* Do not assume an SNR of 22 dB for this part)
$R = 40$kbps implies $b = 5$. So, a 32 CR-QAM constellation has to be used. For this, we need,

$$\overline{P}_e \;\le\; 2 \cdot Q\left(\sqrt{\frac{3 \cdot SNR}{\frac{31}{32}M - 1}}\right) \;<\; 10^{-6}$$

$$\begin{aligned} \frac{3 \cdot SNR}{\frac{31}{32}M - 1} &= 13.8 \text{ dB} \\ SNR &= 13.8 + 10 \;=\; 23.8 \text{ dB} \;=\; 240 \\ \bar{\varepsilon}_x &= 240 \cdot \sigma^2 \\ P_x &= \bar{\varepsilon}_x \cdot N/T \;=\; 240 \cdot \sigma^2 * 2 * 8 \cdot 10^3 \\ &= 3840 * 10^3 \cdot \sigma^2 \;=\; 65.8 + (\sigma^2)_{in\ dB} \text{ dB} \end{aligned}$$

This transmit power maintains the same $\overline{P}_e$.

(d) **(1 pt)** For a QAM system (SQ or CR), to increase b by 2, one needs 6 dB of extra transmit power.
So, a 16 QAM system having an SNR of 22 dB (as seen in part a) has the same P_e as a 64 QAM system having an SNR of 28 dB ($P_e \le 1.8 \cdot 10^{-8}$).
Similarly, a 32 CR-QAM system having an SNR of 23.8 dB (as seen in part c) has the same P_e as a 128 QAM system having an SNR of 29.8 dB ($P_e \le 10^{-6}$).
Thus, at the increased SNR, a 64 QAM system satisfies the P_e requirement, but a 128 CR-QAM system does not. Therefore, the highest data rate that can be reliably sent at the new SNR is $R = 6 * 8k = 48$ kbps.

6. *(1.17) Frequency separation in FSK.* (Adapted from *Wozencraft & Jacobs.*)

   (a) **(2 pts)** First, we have to find the distance between $x_0(t)$ and $x_1(t)$. Before we do this, let us check whether these signals are orthogonal for this particular value of $\Delta = 10^4$.

$$\int_0^T x_0(t)x_1(t)dt = \frac{2\mathcal{E}_x}{T}\int_0^T \cos(2\pi f_0 t)cos(2\pi(f_0+\Delta)t)dt$$

   Using the trigonometric identity, $\cos a \cos b = \frac{1}{2}[\cos(a+b)+\cos(a-b)]$, we obtain,

$$\begin{aligned}\int_0^T x_0(t)x_1(t)dt &= \frac{\mathcal{E}_x}{T}\int_0^T [\cos(2\pi(2f_0+\Delta)t)+\cos 2\pi\Delta t]dt \\ &= \frac{\mathcal{E}_x}{2\pi T}\left[\frac{\sin(2\pi(2f_0+\Delta)T)}{2f_0+\Delta}+\frac{\sin 2\pi\Delta T}{\Delta}\right] \\ &= \frac{\mathcal{E}_x}{2\pi T}\sin 2\pi\Delta T\left[\frac{1}{2f_0+\Delta}+\frac{1}{\Delta}\right],\end{aligned}$$

   since $\sin(2\pi(2f_0+\Delta)T) = \sin 2\pi\Delta T$ (f_0T is an integer).
   For $\Delta = 10^4$, the above quantity is equal to zero, and therefore, the signals are orthogonal. The energy of each signal is $\mathcal{E}_x$, and the distance between the signals is then $\sqrt{2\mathcal{E}_x}$. The corresponding probability of error is then,

$$P_e = Q\left(\frac{\sqrt{2\mathcal{E}_x}}{2\sigma}\right) = Q(\frac{\sqrt{2\times 0.32}}{2\times 0.1}) = Q(4) \simeq 3.2\times 10^{-5}$$

   (b) **(3 pts)** From part (a), the inner product of $x_0(t)$ and $x_1(t)$ is given by,

$$\int_0^T x_0(t)x_1(t)dt = \frac{\mathcal{E}_x}{2\pi T}\sin 2\pi\Delta T\left[\frac{1}{2f_0+\Delta}+\frac{1}{\Delta}\right]$$

   This quantity is equal to zero if and only if $\sin 2\pi\Delta T = 0$, or equivalently if and only if $|\Delta| = \frac{n}{2T}$, $n = 1, 2, \ldots$. Therefore, the smallest Δ is, $\Delta = 5\times 10^3$.
   This constellation is block orthogonal.

7. *(1.21) Comparing Bounds.*

   (a) **(2 pts)** The signal constellation is shown in the figure below.

   From the signal constellation, we get $d_{min} = 2$. Since we have 5 signals, the Union Bound is given by,

$$\begin{aligned}P_e &\leq 4\,Q\left(\frac{d_{min}}{2\sigma}\right) \\ &= 4\,Q\left(\frac{1}{\sigma}\right).\end{aligned}$$

   (b) **(2 pts)** The number of Nearest Neighborhood $N_e = 1/5(2+2+3+3+2) = 2.4$. Therefore, the Nearest Neighborhood Union Bound is given by,

$$P_e = 2.4\,Q\left(\frac{1}{\sigma}\right).$$

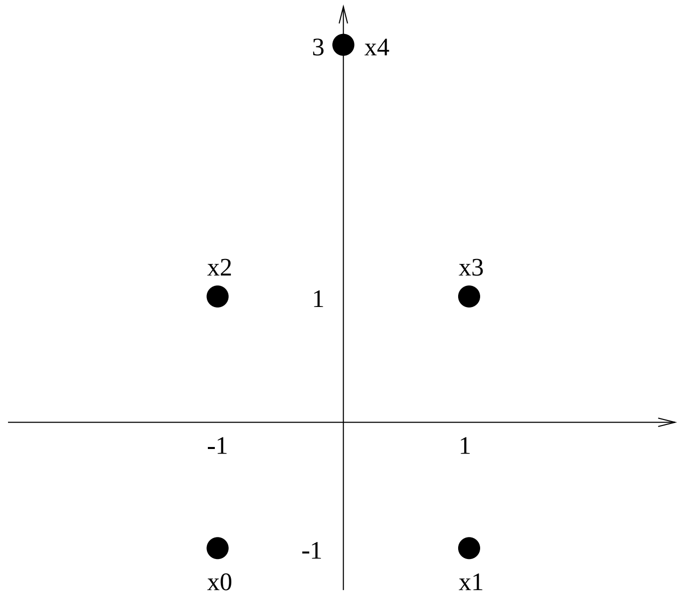

Figure 5: Signal Constellation

(c) **(1 pt)** Since $\mathcal{E}_x = 1/5(2 \times 4 + 3^2) \ = 1.7$,

$$\sigma = \sqrt{\frac{\bar{\mathcal{E}}_x}{SNR}} = \sqrt{\frac{1.7}{10^{1.4}}}$$

Therefore, we get

$$P_e(\text{NNUB}) = 2.4 \times Q\left(\sqrt{\frac{10^{1.4}}{1.7}}\right) = 1.45 \times 10^{-4}.$$

8. *(1.28) Basic Detection*

(a) **(2 pts)** Using the formulae given for $\tilde{Q}(x)$, we get:

$$\begin{aligned}\tilde{Q}(-\infty) &= 1\\ \tilde{Q}(0) &= \frac{1}{2}\\ \tilde{Q}(\infty) &= 0\\ \tilde{Q}(\sqrt{10}) &= 5.7 \cdot 10^{-3}\end{aligned}$$

(b) **(1 pt)** We have:

$$\tilde{Q}(x) = 10^{-6} \Leftrightarrow \frac{1}{2}\exp(-\sqrt{2}x) = 10^{-6} \Leftrightarrow x = 9.3 = 19 \ dB$$

(c) **(2 pts)** Assuming equally likely inputs, we obtain:

$$P_e = P\left(n > \frac{d}{2}\right) = \int_{\frac{d}{2}}^{\infty} \frac{1}{\sigma\sqrt{2}} \exp(-\sqrt{2}\frac{|u|}{\sigma}) = \tilde{Q}(\frac{d}{2\sigma})$$

(d) **(2 pts)** Since $\overline{\mathcal{E}}_x = \frac{d^2}{4}$,

$$SNR = \frac{d^2}{4\sigma^2}$$

and

$$P_e = \tilde{Q}\left(\sqrt{SNR}\right)$$

(e) **(2 pts)** We now consider the case of PAM transmission. We modify the expression holding for Gaussian noise, and we find that:

$$P_e = 2\left(1 - \frac{1}{M}\right)\tilde{Q}\left(\sqrt{\frac{3SNR}{M^2-1}}\right)$$

(f) **(2 pts)** The following table provides the required SNR values for $\bar{b} = 1, 2, 3$ and $P_e = 10^{-6}$.

$\bar{b}$	M	SNR(dB)
1	2	20
2	4	27
3	8	33

(g) **(1 pt)** Comparing the SNR values found above with the SNR values for Gaussian noise cited in the course reader, we conclude that Gaussian noise is preferable.