Homework #5 (77+1 pts)
Solutions

1. *(3.3) The 379 channel model.*

(a) **(1 pt)** The pulse response p(t) is, by definition,

$$\begin{aligned} p(t) &= \varphi(t) * h(t) \\ &= \frac{1}{\sqrt{T}} sinc(\frac{t}{T}) * [\delta(t) - \frac{1}{2}\delta(t-T)] \\ &= \frac{1}{\sqrt{T}}[sinc(\frac{t}{T}) - \frac{1}{2} sinc(\frac{t-T}{T})] \end{aligned}$$

(b) **(2 pts)** The magnitude of p(t), is,

$$\begin{aligned} ||p||^2 &= \int_{-\infty}^{\infty} [\varphi^2(t) + \frac{1}{4}\varphi^2(t-T) - \varphi(t)\varphi(t-T)]dt \\ &= 1 + \frac{1}{4} + 0 \\ &= \frac{5}{4}. \end{aligned}$$

and,

$$\varphi_p(t) = \frac{p(t)}{||p||} = \frac{2}{\sqrt{5T}}[sinc(\frac{t}{T}) - \frac{1}{2} sinc(\frac{t-T}{T})].$$

(c) **(2 pts)** We have,

$$\begin{aligned} q(t) &= \varphi_p(t) * \varphi_p^*(-t) \\ &= \frac{1}{5T}[2sinc(\frac{t}{T}) - sinc(\frac{t-T}{T})] * [2sinc(-\frac{t}{T}) - sinc(\frac{-t-T}{T})] \\ &= \frac{1}{5T}[4T sinc(\frac{t}{T}) - 2T sinc(\frac{t+T}{T}) - 2T sinc(\frac{t-T}{T}) + T sinc(\frac{t}{T})] \\ &= sinc(\frac{t}{T}) - \frac{2}{5} sinc(\frac{t+T}{T}) - \frac{2}{5} sinc(\frac{t-T}{T}). \end{aligned}$$

And, $q(0) = 1 - \frac{2}{5}(0+0) = 1$, $q(-t) = q(t)$, since $sinc(x) = sinc(-x)$.

(d) **(2 pts)** By sampling at $t = kT$, we obtain, $q_k = \delta_k - \frac{2}{5}\delta_{k-1} - \frac{2}{5}\delta_{k+1}$. Therefore 2 symbols are distorted by a given symbol and they will be decreased assuming the given symbol is positive. The plot for $q(t)$ is shown in Figure 1.

(e) **(2 pts)** For 8 PAM, we have $|x|_{max} = 3\frac{1}{2}$. Therefore, the peak distortion $\mathcal{D}_p$ is,

$$\mathcal{D}_p = |x|_{max}||p|| \sum_{k \neq 0} |q_k| = \frac{7}{2} \times \sqrt{\frac{5}{4}} \times \frac{4}{5} = \frac{7}{5}\sqrt{5}$$

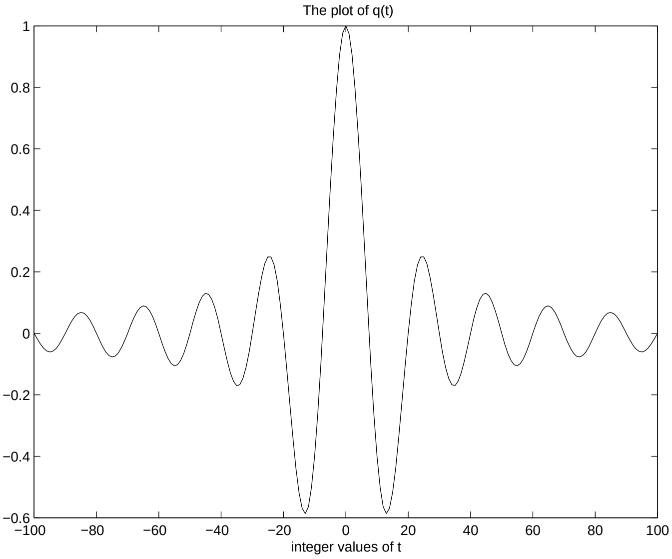

Figure 1: Plot of q(t)

(f) **(2 pts)** The MSE distortion is,

$$\begin{aligned}\mathcal{D}_{MS} &= \mathcal{E}_x ||p||^2 \sum_{k \neq 0} |q_k|^2 \\ &= \frac{21}{4} \cdot \frac{5}{4} \cdot \frac{8}{25} = \frac{21}{10}\end{aligned}$$

where $\mathcal{E}_x = \frac{d^2}{12}(M^2 - 1) = \frac{21}{4}$.
For this case, $\mathcal{D}_p = 1.49 \mathcal{D}_{MS}$.

(g) **(1 pt)** The probability of error can be approximated by,

$$P_e \simeq 1.75 Q\left(\frac{\sqrt{5}}{4\sqrt{\sigma^2 + 2.1}}\right)$$

2. *(3.4) Bias and SNR.*

In this problem, we substitute $\frac{1}{\alpha}$ with β for brevity.

(a) **(1 pt)** The receiver is unbiased if $E[\beta y_k | x_k] = x_k$,

$$\begin{aligned}E[\beta y_k | x_k] &= E[\beta ||p|| x_k + \beta n_k + \beta ||p|| \sum_{m \neq k} x_m q_{k-m} | x_k] \\ &= \beta ||p|| x_k\end{aligned}$$

The receiver is unbiased if $\beta = \frac{1}{||p||} = \frac{2}{\sqrt{5}}$. Equivalently, $\alpha = ||p||$.

(b) **(2 pts)** It is assumed that the signal x_k and the noise n_k are uncorrelated. In this case,

$$\begin{aligned}E e_k^2 &= E[(\beta y_k - x_k)^2] \\ &= E\left[(\beta ||p|| x_k + \beta n_k + \beta ||p|| \sum_{m \neq k} x_m q_{k-m} - x_k)^2\right]\end{aligned}$$

$$= (\beta||p||-1)^2 Ex_k^2 + \beta^2 En_k^2 + \beta^2||p||^2\mathcal{E}_x \sum_{m\neq k} q_{k-m}^2$$
$$= (\beta||p||-1)^2\mathcal{E}_x + \beta^2\sigma^2 + \beta^2\mathcal{D}_{MS}$$

For $\beta = \frac{1}{||p||}$, the MSE becomes, $Ee_k^2 \simeq 1.76$. The corresponding SNR is, $SNR_u = 2.98 = 4.75dB$.

(c) **(2 pts)** From part (b), the MSE is given by,

$$Ee_k^2 = (\beta||p||-1)^2\mathcal{E}_x + \beta^2\sigma^2 + \beta^2\mathcal{D}_{MS}$$

Differentiating with respect to β and setting the expression to zero, yields,

$$\begin{aligned}
\frac{dE[e_k^2]}{d\beta} &= 2(\beta||p||-1)||p||\mathcal{E}_x + 2\beta\sigma^2 + 2\beta\mathcal{D}_{MS} = 0 \\
\Leftrightarrow \beta &= \frac{||p||\mathcal{E}_x}{||p||^2\mathcal{E}_x + \sigma^2 + \mathcal{D}_{MS}} \\
\beta &= \frac{1}{||p||}\left(\frac{1}{1+\frac{1}{SNR_u}}\right)
\end{aligned}$$

You can verify that this is a minimum by showing that $\frac{d^2E[e_k^2]}{d\beta^2} > 0$.
Numerically,

$$\begin{aligned}
\beta &\simeq 0.670 \\
\alpha &= 1.493 \\
SNR_{MMSE} &= 3.98 = 6.0dB
\end{aligned}$$

(d) **(1 pt)** From part (a), $E[\beta y_k|x_k] = \beta||p||x_k = 0.749x_k \neq x_k$. Hence the receiver is biased.

(e) **(1 pt)** By definition,

$$SNR_{MFB} = \frac{\bar{\mathcal{E}}_x||p||^2}{\sigma^2} = 65.625 = 18.2dB.$$

(f) **(1 pt)** In this particular case,

$$SNR_{MFB} > SNR_{MMSE} > SNR_u$$

In general, we will always have $SNR_{MFB} > SNR_u$ and $SNR_{MMSE} > SNR_u$. However we need not have $SNR_{MFB} > SNR_{MMSE}$.

3. *(3.5) Raised cosine pulses with Matlab.*

(a) **(2 pts)** The raised cosine pulse is given by,

$$q(t) = \frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}} \cdot \left[\frac{\cos(\frac{\alpha\pi t}{T})}{1-(\frac{2\alpha t}{T})^2}\right]$$

Division by zero occurs at $t = 0, \frac{T}{2\alpha}, -\frac{T}{2\alpha}$.

- At $t = 0$, q(0)=1.

- At $t = \pm\frac{T}{2\alpha}$. We use L'Hopital's rule,

$$\begin{aligned}
\frac{\cos(\frac{\alpha\pi t}{T})}{1-(\frac{2\alpha t}{T})^2} &= \frac{\frac{d}{dt}\cos(\frac{\alpha\pi t}{T})}{\frac{d}{dt}[1-(\frac{2\alpha t}{T})^2]} \\
&= \frac{\frac{\alpha\pi}{T}\sin(\frac{\alpha\pi t}{T})}{2(\frac{2\alpha t}{T})\frac{2\alpha}{T}} \\
&= \frac{\pi}{4}.
\end{aligned}$$

  Therefore, $q(\frac{T}{2\alpha}) = q(-\frac{T}{2\alpha}) = \frac{\pi}{4}sinc(\frac{1}{2\alpha})$.

(b) **(2 pts)** The plots are shown in Figures 2, 3 and 4. For zero excess bandwidth, we see an obvious deviation from the ideal case because of the non-infinite length sequence (Gibbs phenomenon). (Note:- More details on this can be found in any standard signals textbook. When we truncate the time domain response to 751 points, original frequency response gets convolved with the corresponding 'Dirichlet kernel' to give Gibb's phenomenon effect. This is not visible in the other cases because the time samples neglected are too small to make any difference.)

(c) **(3 pts)** The plots are shown in Figures 5, 6, and 7. The 50% excess bandwidth has the smallest region above -40 dB. We would have expected the zero excess bandwidth to be the best, but the Gibbs phenomenon degraded its behavior i.e. it has to do with the practical FIR implementation.

(d) **(2 pts)** The plots are shown in Figures 8, 9, and 10. As excess bandwidth increases, ISI performance becomes more robust to incorrect sampling.

4. *(3.6) Noise enhancement: MMSE-LE vs ZFE.*

(a) **(2 pts)** The ZFE is given by,

$$W_{ZFE} = \frac{1}{||p||Q(D)} = \frac{||p||}{a^*D^{-1} + ||p||^2 + aD} = \frac{\sqrt{1+aa^*}}{a^*D^{-1} + (1+aa^*) + aD}$$

The MMSE-LE equalizer is,

$$\begin{aligned}
W_{MMSE-LE} &= \frac{1}{||p||(Q(D) + \frac{1}{SNR_{MFB}})} = \frac{||p||}{a^*D^{-1} + ||p||^2(1 + \frac{1}{SNR_{MFB}}) + aD} \\
&= \frac{||p||}{a^*D^{-1} + b + aD}
\end{aligned}$$

where $b = ||p||^2(1 + \frac{1}{SNR_{MFB}})$.

(b) **(6 pts)** By substituting $D = e^{-jw}$, and for a real, the above filters become,

$$\begin{aligned}
W_{ZFE} &= \frac{||p||}{||p||^2 + 2a\cos w} \\
W_{MMSE-LE} &= \frac{||p||}{b + 2a\cos w}
\end{aligned}$$

The plots for a=0.5, 0.9 are shown in Figure 11. From the plots, it is seen that the ZFE produces more noise enhancement at high frequencies. The effect is more noticeable for a=0.9.

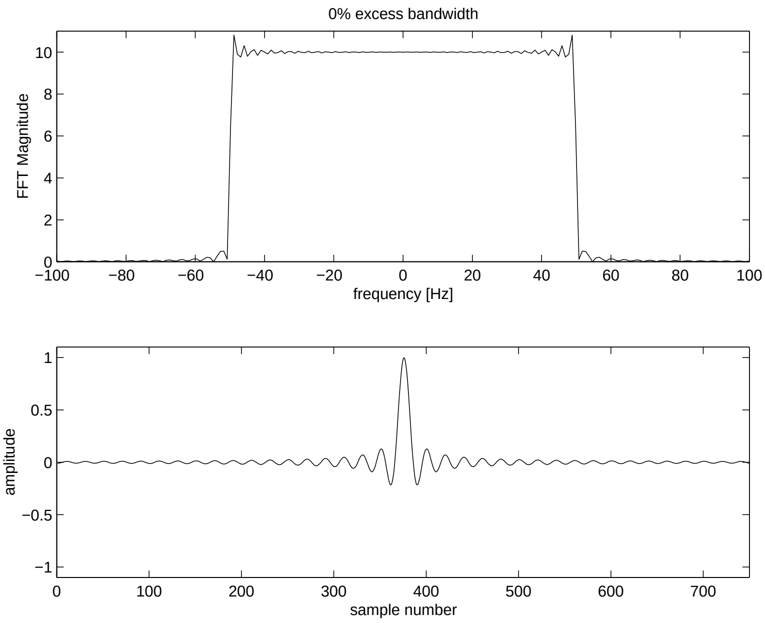

Figure 2: Raised cosine pulse with 0% excess bandwidth.

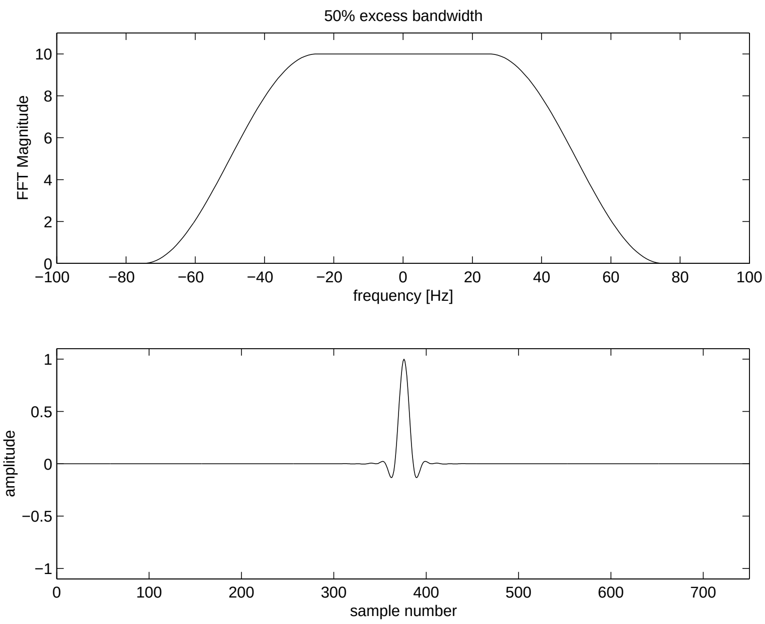

Figure 3: Raised cosine pulse with 50% excess bandwidth.

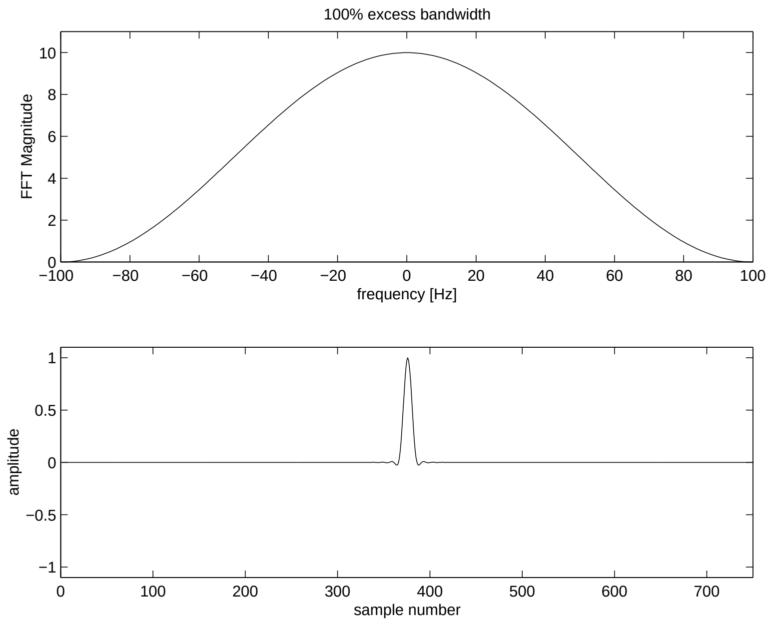

Figure 4: Raised cosine pulse with 100% excess bandwidth.

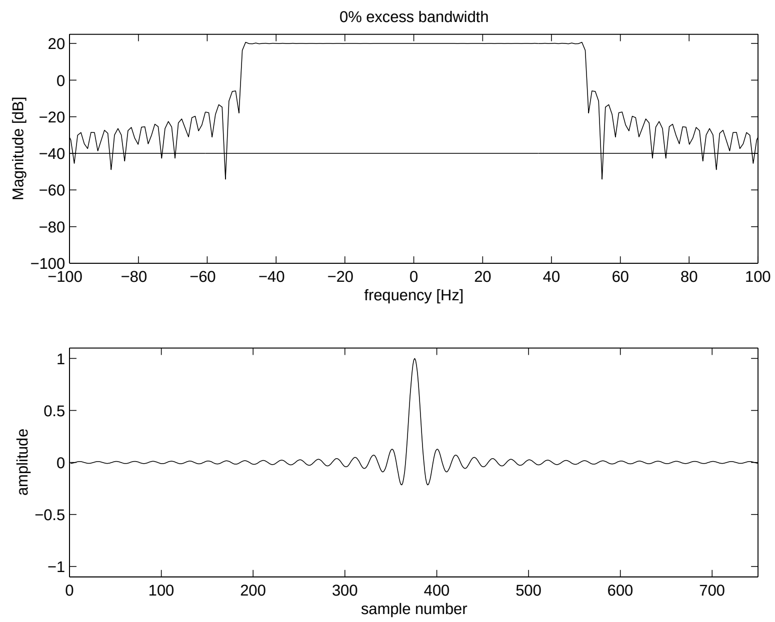

Figure 5: Raised cosine pulse with 0% excess bandwidth (dB scale).

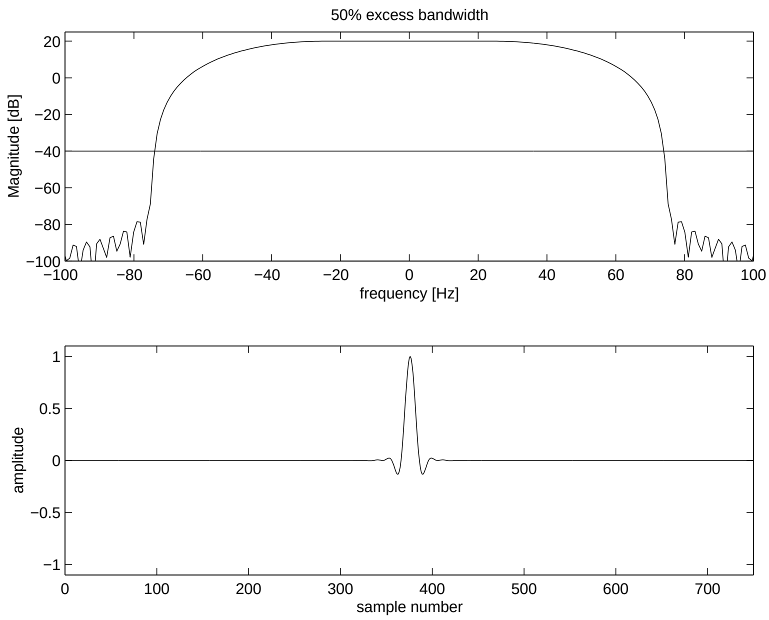

Figure 6: Raised cosine pulse with 50% excess bandwidth (dB scale).

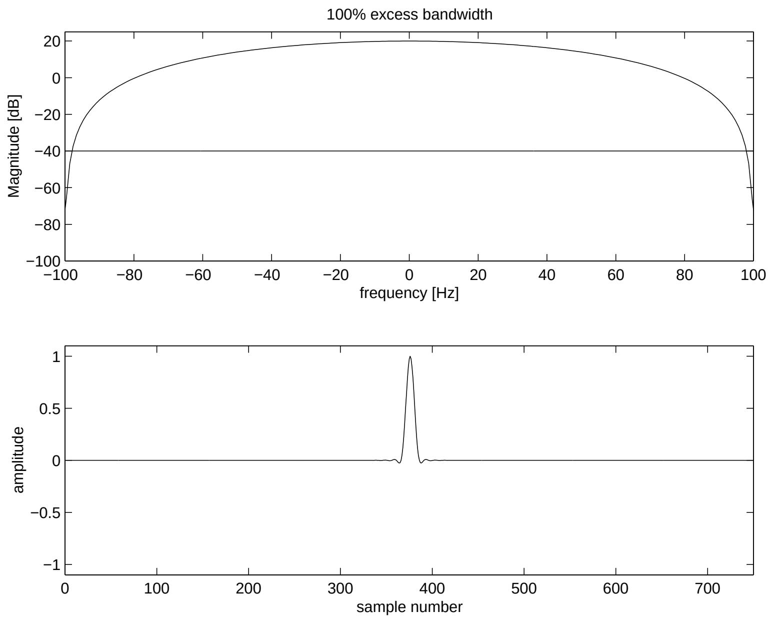

Figure 7: Raised cosine pulse with 100% excess bandwidth (dB scale).

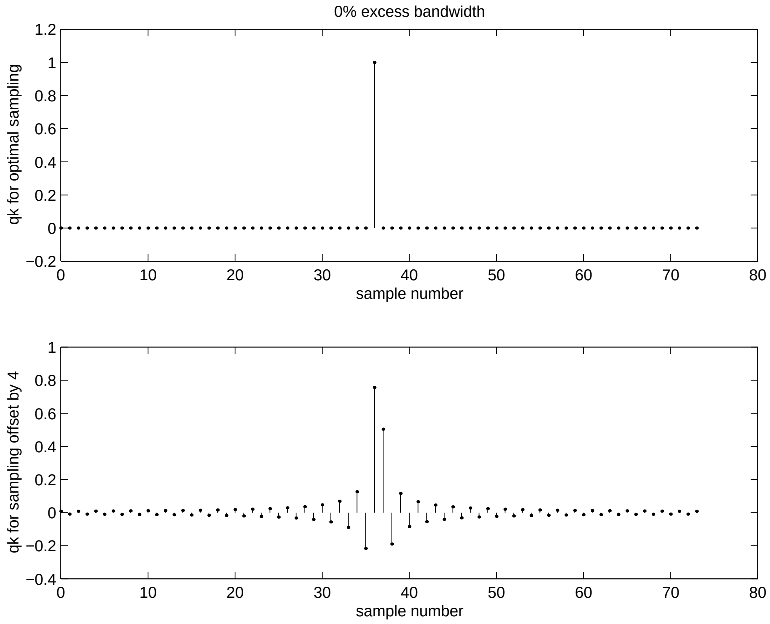

Figure 8: Incorrectly sampled raised cosine pulse with 0% excess bandwidth.

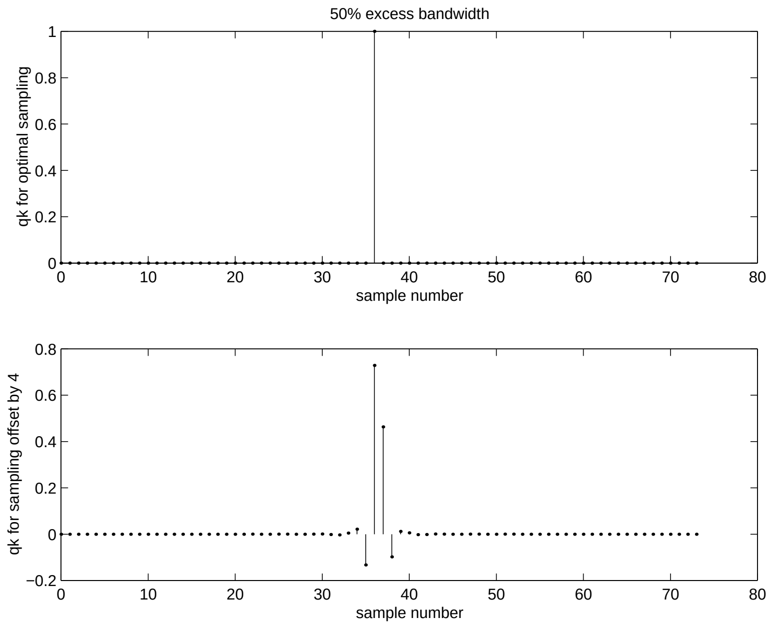

Figure 9: Incorrectly sampled raised cosine pulse with 50% excess bandwidth.

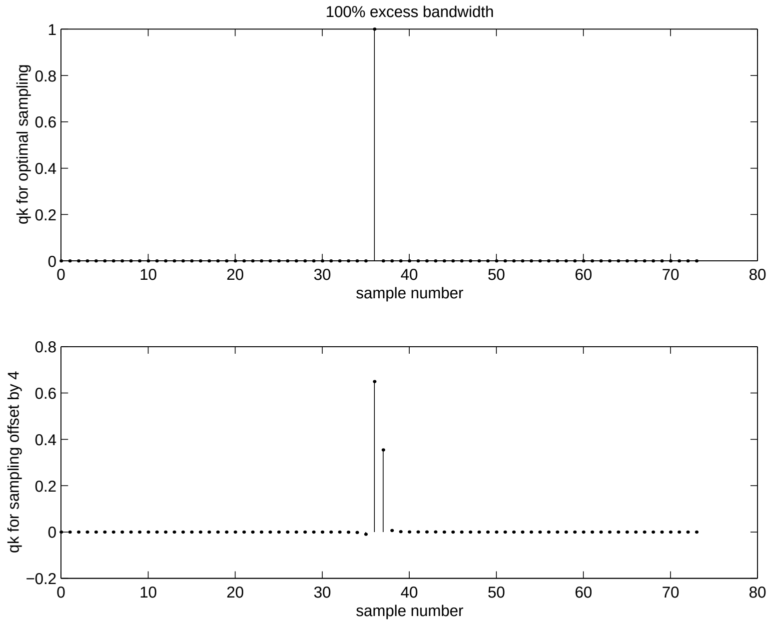

Figure 10: Incorrectly sampled raised cosine pulse with 100% excess bandwidth.

(c) **(3 pts)** The roots r_1, r_2 of the equation $aD^2 + bD + a^* = 0$ are given by,

$$
\begin{aligned}
r_1 &= \frac{-b - \sqrt{b^2 - 4aa^*}}{2a} \\
r_2 &= \frac{-b + \sqrt{b^2 - 4aa^*}}{2a}
\end{aligned}
$$

Since b and aa^* are real, $b^2 - 4aa^*$ is real. To prove that it is always positive,

$$
\begin{aligned}
b^2 - 4aa^* &= [||p||^2(1 + \frac{1}{SNR_{MFB}})]^2 - 4aa^* \\
&\geq [||p||^2]^2 - 4aa^* = (1 - aa^*)^2 > 0
\end{aligned}
$$

Finally, $r_1 r_2^*$ is computed,

$$
\begin{aligned}
r_1 r_2^* &= \frac{1}{4aa^*}(-b - \sqrt{b^2 - 4aa^*})(-b + \sqrt{b^2 - 4aa^*}) \\
&= \frac{1}{4aa^*}(b^2 - (\sqrt{b^2 - 4aa^*})^2) \\
&= 1
\end{aligned}
$$

(d) **(2 pts)** For the MMSE-LE, the filter is written as,

$$
W_{MMSE-LE} = \frac{||p||D}{a^* + bD + aD^2} = \frac{||p||}{a} \cdot \frac{D}{(D - r_1)(D - r_2)}
$$

By expanding $W_{MMSE-LE}$ into partial fractions,

$$
W_{MMSE-LE}(D) = \frac{||p||}{a}\left[\frac{A}{D - r_1} + \frac{B}{D - r_2}\right]
$$

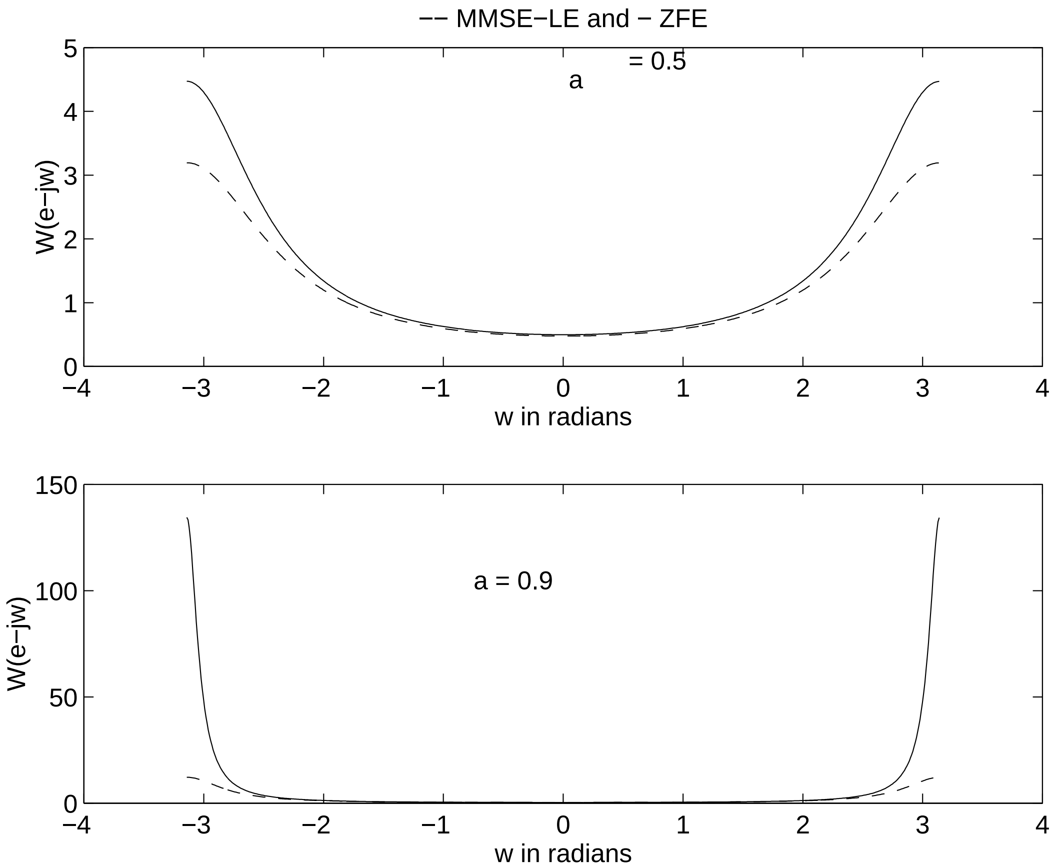

Figure 11: Plot of $W(e^{jw})$ for ZFE and MMSE-LE.

We find that $A = \frac{r_1}{r_1 - r_2}$ and $B = \frac{r_2}{r_2 - r_1}$. So,

$$W_{MMSE-LE}(D) = \frac{||p||}{a(r_1 - r_2)} \left[\frac{r_1}{D - r_1} - \frac{r_2}{D - r_2} \right]$$

(e) **(2 pts)** Assuming that the sequences are stable, we have,

$$\frac{r}{D - r} \leftrightarrow \begin{cases} -(\frac{1}{r})^n u(n) & \text{for } |r| > 1 \\ (\frac{1}{r})^n u(-n-1) & \text{for } |r| < 1 \end{cases}$$

Evaluating at time $n = 0$,

$$\begin{aligned} -(\frac{1}{r})^n u(n) &= -1, \qquad \text{for } n = 0 \\ (\frac{1}{r})^n u(-n-1) &= 0, \qquad \text{for } n = 0 \end{aligned}$$

Therefore, by taking the inverse D transform of $W_{MMSE-LE}(D)$ and evaluating at $n = 0$, noting that $|r_1| > |r_2|$ so that $|r_1| > 1$, $|r_2| < 1$, yields,

$$w_0 = \frac{||p||}{a(r_1 - r_2)} [-1 - 0]$$

Replacing for r_1 and r_2, gives, after simplification, $w_0 = \frac{||p||}{\sqrt{b^2-4aa^*}}$.

For the ZFE, we simply take $\frac{1}{SNR_{MFB}} = 0$, which leads to, $b = ||p||^2$, and $w_0 = \frac{||p||}{1-aa^*}$

(f) **(4 pts)** The noise variance σ^2_{ZFE} for the ZFE equalizer, is by definition,

$$\sigma^2_{ZFE} = \frac{w_0\sigma^2}{||p||} = \frac{\sigma^2}{1-aa^*}$$

The corresponding loss γ_{ZFE} is,

$$\gamma_{ZFE} = \frac{||p||^2\sigma^2_{ZFE}}{\sigma^2} = \frac{1+aa^*}{1-aa^*}$$

Similarly,

$$\sigma^2_{MMSE-LE} = \frac{w_0\sigma^2}{||p||} = \frac{\sigma^2}{\sqrt{b^2-4aa^*}}$$

The loss with respect to SNR_{MFB} is given by,

$$\begin{aligned}\gamma_{MMSE-LE} &= \frac{||p||^2\sigma^2_{MMSE-LE}}{\sigma^2}\frac{\overline{\mathcal{E}}_x}{\overline{\mathcal{E}}_x - \sigma^2_{MMSE-LE}} \\ &= \frac{(1+aa^*)\overline{\mathcal{E}}_x}{\overline{\mathcal{E}}_x\sqrt{b^2-4aa^*} - \sigma^2}.\end{aligned}$$

For the particular case $\overline{\mathcal{E}}_x = 1$ and $\sigma^2 = 0.1$, $b^2 - 4aa^*$ is given by,

$$b^2 - 4aa^* = 1.21 - 1.8aa^* + a^2(a^*)^2$$

(g) **(4 pts)** The losses with respect to SNR_{MFB} were computed for $a = 0, 0.5, 1$. The results are:

a	γ_{ZFE} (dB)	$\gamma_{MMSE-LE}$ (dB)
0.0	0.00	0.00
0.5	2.22	1.90
1.0	∞	5.68

The plots for γ_{ZFE} and $\gamma_{MMSE-LE}$ are shown in Figure 12.

5. *(3.10) Bias and Probability of Error.*

(a) **(1 pt)** The MSE is given by,

$$E[{e_k}^2] = E(y_k - x_k)^2] = E[n_k^2] = 0.1.$$

(b) **(1 pt)** The probability of error is given by,

$$\begin{aligned}P_e &= \frac{1}{3}P_{e|x=-1} + \frac{1}{3}P_{e|x=0} + \frac{1}{3}P_{e|x=1} \\ &= \frac{1}{3}Q\left(\frac{1}{2\sigma}\right) + \frac{2}{3}Q\left(\frac{1}{2\sigma}\right) + \frac{1}{3}Q\left(\frac{1}{2\sigma}\right) \\ &= \frac{4}{3}Q\left(\frac{1}{2\sigma}\right) \\ &= \frac{4}{3}Q\left(\frac{1}{2\sqrt{0.1}}\right) \\ &= \frac{4}{3} \times 0.0565 = 0.0753.\end{aligned}$$

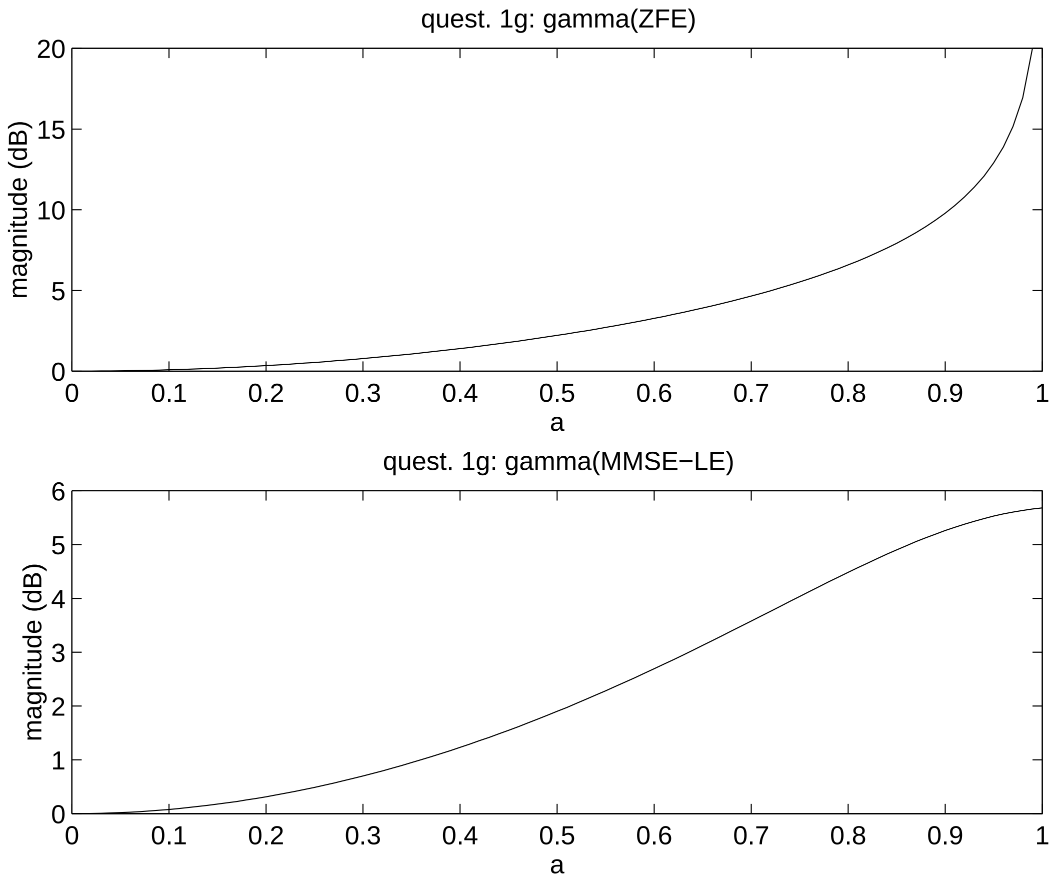

Figure 12: Plot of γ for ZFE and MMSE-LE

(c) **(2 pts)** It is assumed here that the signal x_k and the noise n_k are uncorrelated. The MSE is given by,

$$E[e_k^2] = E[(x_k - \alpha y_k)^2].$$

Differentiating with respect to α and setting the expression to zero, yields,

$$\begin{aligned} \frac{dE[e_k^2]}{d\alpha} &= 2E[(x_k - \alpha y_k) \cdot (-y_k)] = 0 \\ \Longrightarrow \alpha &= \frac{\mathcal{E}_x}{\mathcal{E}_x + \sigma^2} = \frac{2}{2.3} \end{aligned}$$

You can verify that this is a minimum by showing that

$$\frac{d^2 E[e_k^2]}{d\alpha^2} = 2E[y_k^2] > 0$$

(d) **(2 pts)** With the α found in part(c), we can get

$$\begin{aligned} E[e_k^2] &= E[(x_k - \alpha y_k)^2] \\ &= E[((1-\alpha)x_k - \alpha n_k)^2] \end{aligned}$$

$$\begin{aligned}
&= (1-\alpha)^2\mathcal{E}_x + \alpha^2\sigma^2 \\
&= \left(1 - \frac{2}{2.3}\right)^2 \frac{2}{3} + \left(\frac{2}{2.3}\right)^2 0.1 \\
&= 0.087.
\end{aligned}$$

You can think of αy_k as having the signal $\tilde{x}_k = \alpha x_k$ and the noise $\tilde{n}_k = \alpha n_k$. Then the signal constellation of $\tilde{x}_k$ becomes $\{-\alpha, 0, \alpha\}$ and $\tilde{n}_k$ has the variance $(\alpha\sigma)^2$. Therefore, the probability of error using the same decision boundary in part (b) is given by,

$$\begin{aligned}
P_e &= \frac{1}{3}P_{e|\tilde{x}=-\alpha} + \frac{1}{3}P_{e|\tilde{x}=0} + \frac{1}{3}P_{e|\tilde{x}=\alpha} \\
&= \frac{1}{3}Q\left(\frac{-1/2+\alpha}{\alpha\sigma}\right) + \frac{2}{3}Q\left(\frac{1}{2\alpha\sigma}\right) + \frac{1}{3}Q\left(\frac{\alpha-1/2}{\alpha\sigma}\right) \\
&= \frac{2}{3}Q\left(\frac{.85}{2\sqrt{0.1}}\right) + \frac{2}{3}Q\left(\frac{1.15}{2\sqrt{0.1}}\right) \\
&= \frac{2}{3} \times 0.087 + \frac{2}{3} \times 0.0343 \\
&= 0.082
\end{aligned}$$

(e) **(1 pt)** The optimal decision rule(ML detector) is,

$$\hat{x} = \begin{cases} -1 & \text{when } \tilde{y}_k = \alpha y_k < -\frac{\alpha}{2} \\ 1 & \text{when } \tilde{y}_k = \alpha y_k > \frac{\alpha}{2} \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that the probability of error is the same as that of part(b), either by direct calculation or by noting that the signal part and the noise part are scaled by the same factor.

6. *(3.21) ISI quantification.*
The first step is to obtain a discrete-time ISI channel model(ie. $Q(D)$) from the continuous-time channel $P(\omega)$. To this end, consider

$$\begin{aligned}
P(f) &= \sqrt{T}(1 + 0.9 \cdot e^{j2\pi fT}) \cdot \sqcap(fT) \qquad \text{, where} \\
\sqcap(fT) &= \begin{cases} 1 & \text{, when } \quad |f| < \frac{1}{2T} \\ 0 & \text{, else} \end{cases}
\end{aligned}$$

Therefore, we calculate

$$\begin{aligned}
p(t) &= F^{-1}\{P(f)\} \\
&= \sqrt{T}[\delta(t) + 0.9\delta(t+T)] * [\frac{1}{T}sinc(\frac{t}{T})] \\
&= \frac{1}{\sqrt{T}}[sinc(\frac{t}{T}) + 0.9 \cdot sinc(\frac{t+T}{T})] \\
p^*(-t) &= \frac{1}{\sqrt{T}}[sinc(\frac{t}{T}) + 0.9 \cdot sinc(\frac{t-T}{T})]
\end{aligned}$$

since sinc(t) is an even function. Therefore,

$$\begin{aligned} p(t) * p^*(-t) &= 1.81 \cdot sinc(\frac{t}{T}) + 0.9 \cdot sinc(\frac{t+T}{T}) + 0.9 \cdot sinc(\frac{t-T}{T}) \\ \|p\|^2 &= \int_{-\infty}^{\infty} |p(t)|^2 \, dt \quad = \quad [p(t) * p^*(-t)]_{t\,=\,0} \quad = \quad 1.81 \end{aligned}$$

where we've used the fact that $sinc(t/T - n) * sinc(t/T - m) = T \cdot sinc(t/T - m - n)$ (this is easy to see in the Fourier domain).

$$\begin{aligned} q(t) &= \frac{1}{\|p\|^2} p(t) * p^*(-t) \\ &= sinc(\frac{t}{T}) + \frac{0.9}{1.81} \cdot sinc(\frac{t+T}{T}) + \frac{0.9}{1.81} \cdot sinc(\frac{t-T}{T}) \end{aligned}$$

Thus, by sampling $q(t)$ at $t = kT$, we get

$$\begin{aligned} q_k &= \delta_k + \frac{0.9}{1.81}\delta_{k+1} + \frac{0.9}{1.81}\delta_{k-1} \\ Q(D) &= \frac{0.9}{1.81}D^{-1} + 1 + \frac{0.9}{1.81}D \end{aligned}$$

This gives us the discrete-time ISI channel model. Note that you could have also use your own favourite analog to digital conversion formulae to do this conversion.

Now that the hard work is done, the problem can be solved quickly.

(a) **(1 pt)** The peak distortion is

$$\begin{aligned} D_p &= \|p\| \cdot |x|_{max} \sum_{k\,\neq\,0} |q_k| \\ &= \sqrt{1.81} \cdot 1 \cdot (2 \cdot 0.9/1.81) \quad = \quad 1.338 \end{aligned}$$

(b) **(2 pts)** $SNR_{MFB} = \overline{E}_x \cdot \|p\|^2/\sigma^2 = 10$, and so $\sigma^2 = 0.181$.

$$\begin{aligned} P_e &= Q(\frac{d_{min}\|p\|/2 - D_p}{\sigma}) \\ &= Q(\frac{2 \cdot \sqrt{1.81}/2 - 1.338}{\sqrt{0.181}}) \\ &= Q(0.01747) \quad = \quad 0.493 \qquad \text{(or 0.5 if you don't have Matlab !)} \end{aligned}$$

(c) **(1 pt)** The mean-square distortion is

$$\begin{aligned} D_{ms} &= \|p\|^2 \cdot E_x \sum_{k\,\neq\,0} |q_k|^2 \\ &= 1.81 \cdot 1 \cdot (2 \cdot 0.9^2/1.81^2) \quad = \quad 0.895 \qquad \text{which is} < D_p^2 \end{aligned}$$

(d) **(2 pts)** The probability of error is

$$\begin{aligned} P_e &= Q\left(\frac{d_{min}\|p\|/2}{\sqrt{\sigma^2 + D_{ms}}}\right) \\ &= Q\left(\frac{2\sqrt{1.81}/2}{\sqrt{0.181 + 0.895}}\right) \\ &= Q(1.297) \quad = \quad 0.0973 \end{aligned}$$

(e) **(2 pts)** As shown in Example 3.3.1 of the course reader, the SNR for ZFE is:

$$\begin{aligned} SNR_{ZFE} &= \frac{\bar{\mathcal{E}}_x}{5.26\frac{N_0}{2}} \\ &= 0.2\ dB \end{aligned}$$

Also,

$$\begin{aligned} P_{e,ZFE} &\approx Q\left(\sqrt{SNR_{ZFE}}\right) \\ &= 0.153 \end{aligned}$$

In part d we found that:

$$\begin{aligned} SNR &= 1.297^2 = \\ &= 2.2\ dB \end{aligned}$$

The SNR difference is 2 dB, and it is evident that the ZFE performs worse than doing no equalization.

Note the following points:

- Whatever method you use to obtain $Q(D)$ from the analog channel, make sure that $q_0 = 1$.
- D_{ms} dominates the mean-square error in part d. Thus, ISI can severly degrade receiver performance.

7. *(3.30) Equalizers.*

(a) **(1pt)** The pulse response is $p(t) = \phi(t) * h(t)$. In frequency domain,

$$\begin{aligned} P(f) &= \Phi(f)H(f) \\ &= (\sqrt{T} \sqcap (Tf))(\frac{1}{1+ae^{j2\pi f}} \sqcap (f)) \\ &= \frac{1}{1+ae^{j2\pi f}} \sqcap (f) \qquad \text{(since T=1)} \end{aligned}$$

In terms of ω,

$$P(\omega) = \begin{cases} \frac{1}{1+ae^{j\omega}} & |\omega| \le \pi \\ 0 & |\omega| > \pi \end{cases}$$

(b) **(2 pts)** First, let's find $P(e^{-j\omega T})$.

$$P(e^{-j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(\omega + \frac{2\pi n}{T})$$

Since T=1 and $P(\omega) = 0$ for $|\omega| > \pi$, $P(e^{-j\omega T}) = \frac{1}{1+ae^{j\omega}}$. Then, by the inverse Fourier transform,

$$p_k = (-a)^{-k} u[-k].$$

Therefore,

$$\begin{aligned}
\|p\|^2 &= T\sum_{k=-\infty}^{\infty}|p_k|^2 \\
&= \sum_{k=-\infty}^{0}(-a)^{2k} \\
&= \sum_{k=0}^{\infty}(-a)^{2k} \\
&= \frac{1}{1-a^2}.
\end{aligned}$$

(c) **(3 pts)** By substituting $e^{-j\omega T} = D$ into $P(e^{-j\omega T})$, we get $P(D) = \frac{1}{1+aD^{-1}}$. Therefore,

$$Q(D) = \frac{T}{\|p\|^2}P(D)P^*(D^{-*}) = \frac{1-a^2}{(1+aD)(1+aD^{-1})}.$$

(d) **(3 pts)** For the zero-forcing equalizer,

$$W_{ZFE}(D) = \frac{1}{\|p\|Q(D)} = \frac{(1+aD)(1+aD^{-1})}{\sqrt{1-a^2}}.$$

To calculate $W_{MMSE-LE}(D)$, we need SNR_{MFB}:

$$SNR_{MFB} = \frac{\|p\|^2\overline{\mathcal{E}_x}}{\sigma^2} = \frac{10^{1.5}}{1-a^2}.$$

Then,

$$\begin{aligned}
W_{MMSE-LE}(D) &= \frac{1}{\|p\|(Q(D)+1/SNR_{MFB})} \\
&= \frac{\sqrt{1-a^2}}{\frac{1-a^2}{(1+aD)(1+aD^{-1})}+\frac{1+a^2}{10^{1.5}}} \\
&= \frac{(1+aD)(1+aD^{-1})}{\sqrt{1-a^2}[1+(1+aD)(1+aD^{-1})10^{-1.5}]}.
\end{aligned}$$

(e) **(1 pt)** When $a = 0$, $Q(D) = 1$ and $\|p\|^2 = 1$. Since $SNR = 15$dB and $\Gamma = 8.8$dB at $P_e = 10^{-6}$,

$$\bar{b} = \frac{1}{2}\log_2(1+\frac{10^{1.5}}{10^{0.88}}) = 1.18$$

Then, the maximum data rate achievable is

$$R = \frac{b}{T} = \frac{1.18}{1} = 1.18 \text{ bits/sec}$$