# Diversity-Multiplexing Tradeoff Analysis for MMSE-based Cooperative MIMO Relaying Systems

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***Abstract*—In this paper, we present the diversity-multiplexing tradeoff (DMT) analysis for minimum mean squared error (MMSE) based amplify-and-forward cooperative multiple antenna half-duplex relaying systems where a non-negligible direct link exists between the source and the destination. First, we evaluate an upperbound of the DMT which offers a theoretical limit of the system and show that the upperbound is actually achievable by existing optimal and suboptimal designs for the relay amplifying matrix. Thereby, we establish the optimal DMT for the MMSE-based cooperative relaying system. Our analysis also illustrate the optimality of the conventional relay matrix designs in terms of the DMT and leads to several interesting observations. Finally, numerical simulations demonstrate the accuracy of our analysis.**

## I. Introduction

In a recent decade, it has been well recognized that multiple-input and multiple-output (MIMO) wireless systems can improve link performance and spectral efficiency by utilizing diversity and multiplexing gains [1] [2]. Recently, relay cooperative techniques have also garnered a significant interest thanks to the advantages such as extended cell coverage and improved reliability [3]. For this reason, MIMO relaying systems have been considered as a powerful candidate for next generation wireless networks.

In practical relay networks, one of the most popular relaying protocols is amplify-and-forward (AF) due to its simplicity, which amplifies the signal received from the source and forward it to the destination [4] [5]. In AF MIMO relaying systems, designs of the optimum amplifying matrix (or transceiver) at the relay have been active research areas over the past few years. In *pure relaying* channels which do not have a direct link from the source to the destination, many studies have been carried out to maximize the transmit rate as in [6] and references therein. When the decoding complexity is an issue, minimum mean squared error (MMSE) based approaches have also been studied in [7] and [8]. Extending to the *cooperative relaying* channel which contains a non-negligible direct link, the authors in [9] attempted to find the optimal solution for the relay transceiver under the MMSE criterion resorting to an iterative gradient method, because the problem is non-convex. In addition, as a low complexity alternative, a near optimal closed-form solution has recently been proposed in [10].

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No.2010-0017909).

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In fact, important performance measures such as a signal-to-noise ratio (SNR) distribution or an outage behavior of the MMSE strategy have widely been investigated in the literature over point-to-point channels [11] [12]. Also, the authors in [13] have recently characterized the high SNR bit error rate performance of MMSE-based pure relaying systems. However, for cooperative relaying systems, there is no reported work for the analytical performance that can explain numerical observations.

In this paper, we investigate the error performance limit of MMSE-based cooperative relaying systems using a diversity-multiplexing tradeoff (DMT) analysis which provides a compact characterization of the tradeoff between the transmission rate and the diversity order [1], and gives a convenient tool for comparing various relaying systems with different protocols [3] [4]. We first evaluate an upperbound of the DMT which offers a theoretical limit of the system. Then by showing that the upperbound is actually achievable by the optimal and suboptimal relay matrix designs proposed in [9] and [10], respectively, we establish the optimal DMT for MMSE-based cooperative relaying systems. Our analysis illustrates the optimality of existing solutions for the relay matrix design [9] [10] in terms of the DMT and provides a helpful guideline for designing the relaying system under the MMSE criterion. Finally, computer simulations demonstrate the accuracy of our analysis.

Throughout this paper, normal letters represent scalar quantities, boldface letters indicate vectors and boldface uppercase letters designate matrices. We use $\mathbb{S}_{++}^{N}$ to denote a set of $N \times N$ positive definite matrices. The superscript $(\cdot)^H$ stands for conjugate transpose. $\mathbf{I}_N$ and $E[\cdot]$ are defined as an $N \times N$ identity matrix and the expectation operator, respectively. $\mathrm{Tr}\,(\mathbf{A})$ and $[\mathbf{A}]_{k,k}$ denote the trace and the $k$-th diagonal element of a matrix $\mathbf{A}$, respectively.

## II. System Model

In this paper, we consider a cooperative relaying system in Fig. 1 where one AF relay node helps the communication between the source and the destination in the presence of a direct link. The source, relay, and destination nodes are equipped with $N_t$, $N_r$, and $N_d$ antennas, respectively. Our discussion will focus on uncorrelated flat fading relay channels as in [6]–[10] where no channel state information (CSI) is allowed at the source, while both the relay and the destination have perfect CSI of all links. We assume a spatial multiplexing system which transmits $N_t$ data streams simultaneously where $N_t \leq \min(N_r, N_d)$ as in [10]. It is also assumed that each data transmission occurs in two separate time slots due to loop interference in the relay node.

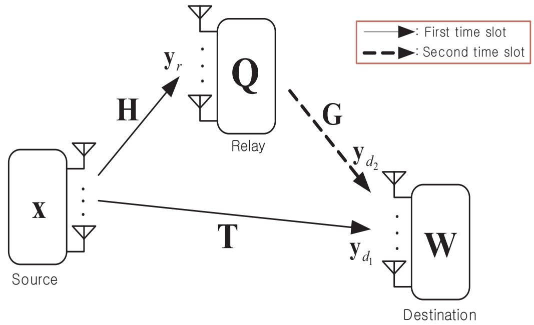


Fig. 1. System description of a MIMO cooperative AF relay network

In the first time slot, the source broadcasts the signal vector $\mathbf{x} \in \mathbb{C}^{N_t \times 1}$ to both the relay and the destination, and the received signals at the relay and at the destination, $\mathbf{y}_r \in \mathbb{C}^{N_r \times 1}$ and $\mathbf{y}_{d_1} \in \mathbb{C}^{N_d \times 1}$, are respectively given by

$$\mathbf{y}_r = \mathbf{H}\mathbf{x} + \mathbf{n}_r \quad \text{and} \quad \mathbf{y}_{d_1} = \mathbf{T}\mathbf{x} + \mathbf{n}_{d_1},$$

where $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ and $\mathbf{T} \in \mathbb{C}^{N_d \times N_t}$ denote the source-to-relay and the source-to-destination (direct link) channel matrices, respectively, and $\mathbf{n}_r \in \mathbb{C}^{N_r \times 1}$ and $\mathbf{n}_{d_1} \in \mathbb{C}^{N_d \times 1}$ indicate the noise vectors at the relay and at the destination, respectively. In this phase, we have a source power constraint as $E[\|\mathbf{x}\|^2] \leq N_t\rho$ where $\rho \triangleq P_T/N_t$ denotes the input signal-to-noise ratio (SNR) and $P_T$ indicates the total source transmit power.

Next, in the subsequent time slot, the relay signal $\mathbf{y}_r$ is precoded by the relay transceiver $\mathbf{Q} \in \mathbb{C}^{N_r \times N_r}$ and transmitted to the destination. Then, the received signal at the destination is written by

$$\mathbf{y}_{d_2} = \mathbf{G}\mathbf{Q}\mathbf{H}\mathbf{x} + \mathbf{n}_{d_2},$$

where $\mathbf{n}_{d_2} \triangleq \mathbf{G}\mathbf{Q}\mathbf{n}_r + \mathbf{n}_d$ designates the effective noise vector in the second time slot with covariance matrix $\mathbf{R}_n \triangleq \mathbf{G}\mathbf{Q}\mathbf{Q}^H\mathbf{G}^H + \mathbf{I}_{N_d}$. In this case, the relay matrix $\mathbf{Q}$ needs to satisfy the relay power constraint $P_R$ as $E[\|\mathbf{Q}\mathbf{y}_r\|^2] \leq P_R$. We assume that all channel matrices have random entries which are independent and identically distributed (i.i.d.) complex Gaussian $\sim \mathcal{CN}(0,1)$, but remain constant over two time slots. All elements of the noise vectors $\mathbf{n}_r$, $\mathbf{n}_{d_1}$ and $\mathbf{n}_d$ are also assumed to be i.i.d. $\sim \mathcal{CN}(0,1)$.

As a result, combining two signals received at the destination over two consecutive time slots, we have the signal vector $\mathbf{y}_d \in \mathbb{C}^{2N_d \times 1}$ at the destination as

$$\mathbf{y}_d = \begin{bmatrix} \mathbf{y}_{d_1} \\ \mathbf{y}_{d_2} \end{bmatrix} = \begin{bmatrix} \mathbf{T} \\ \mathbf{G}\mathbf{Q}\mathbf{H} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{n}_{d_1} \\ \mathbf{n}_{d_2} \end{bmatrix}. \qquad (1)$$

Finally, when a MMSE linear receiver $\mathbf{W} \in \mathbb{C}^{N_t \times 2N_d}$ [9] is employed at the destination, the estimated signal waveform $\mathbf{s} \in \mathbb{C}^{N_t \times 1}$ is expressed as $\mathbf{s} = \mathbf{W}\mathbf{y}_d$.

## III. Preliminaries

In this section, we summarize several important results for designs of the relay amplifying matrix in MMSE-based cooperative relaying systems.

Recently, it has been shown that the MMSE optimal relay matrix $\hat{\mathbf{Q}}$ is generally expressed as $\hat{\mathbf{Q}} = \mathbf{B}\mathbf{L}$ where $\mathbf{B} \in \mathbb{C}^{N_r \times N_t}$ and $\mathbf{L} = (\mathbf{H}^H\mathbf{H} + \mathbf{T}^H\mathbf{T} + \rho^{-1}\mathbf{I}_{N_t})^{-1}\mathbf{H}^H \in \mathbb{C}^{N_t \times N_r}$ represents the relay precoder and receiver, respectively (see [10] for detail). Then, for given linear receivers $\mathbf{W}$ and $\mathbf{L}$, the error covariance matrix denoted by $\mathbf{R}_e \triangleq (\mathbf{s}-\mathbf{x})(\mathbf{s}-\mathbf{x})^H$ can be represented as a sum of two individual error covariance matrices as

$$\mathbf{R}_e = \left(\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1} + \left(\mathbf{B}^H\mathbf{G}^H\mathbf{G}\mathbf{B} + \mathbf{\Omega}^{-1}\right)^{-1}, \qquad (2)$$

where $\mathbf{R}_T \in \mathbb{S}_{++}^{N_t}$ and $\mathbf{\Omega} \in \mathbb{S}_{++}^{N_t}$ are defined as $\mathbf{R}_T \triangleq (\mathbf{T}^H\mathbf{T} + \rho^{-1}\mathbf{I}_{N_t})^{-1}$ and $\mathbf{\Omega} \triangleq \mathbf{L}\mathbf{H}\mathbf{R}_T$, respectively. The first term of $\mathbf{R}_e$ corresponds to the MSE in the broadcast phase in the first time slot and the second term indicates the incremental MSE due to the multiple access phase of the second time slot.

Now, we define two eigenvalue decompositions $\mathbf{G}^H\mathbf{G} = \mathbf{V}_g\mathbf{\Lambda}_g\mathbf{V}_g^H$ and $\mathbf{\Omega} = \mathbf{U}_\omega\mathbf{\Lambda}_\omega\mathbf{U}_\omega^H$ where $\mathbf{\Lambda}_g$ and $\mathbf{\Lambda}_\omega$ represent square diagonal matrices with eigenvalues $\lambda_{\omega,k}$ for $k = 1, \ldots, N_t$ and $\lambda_{g,k}$ for $k = 1, \ldots, N_r$ arranged in descending order. Then, it is also known that the relay precoder $\mathbf{B}$ can be written without loss of any optimality as $\mathbf{B} = \widetilde{\mathbf{V}}_g\mathbf{\Phi}\mathbf{U}_\omega^H$ where $\widetilde{\mathbf{V}}_g$ denotes a matrix constructed by the first $N_t$ columns of a unitary matrix $\mathbf{V}_g$ and $\mathbf{\Phi} \in \mathbb{C}^{N_t \times N_t}$ is an arbitrary matrix. Since the first term of $\mathbf{R}_e$ consists of known parameters, the original joint MMSE optimization problem for $\mathbf{Q}$ and $\mathbf{W}$, i.e., $\min_{\mathbf{Q},\mathbf{W}} \operatorname{Tr}(\mathbf{R}_e)$ reduces to optimizing $\mathbf{\Phi}$ as

$$\begin{aligned} \hat{\mathbf{\Phi}} = \min_{\mathbf{\Phi}} & \operatorname{Tr}\left(\mathbf{\Phi}^H\mathbf{\Lambda}_g\mathbf{\Phi} + \mathbf{\Lambda}_\omega^{-1}\right)^{-1} \\ \textit{s.t.} & \operatorname{Tr}(\mathbf{\Phi}\mathbf{R}_\omega\mathbf{\Phi}^H) \leq P_R, \end{aligned} \qquad (3)$$

where $\mathbf{R}_\omega = \mathbf{U}_\omega^H\mathbf{L}(\rho\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_t})\mathbf{L}^H\mathbf{U}_\omega$.

Problem (3) is non-convex in general, and thus the solution can be found by two different approaches: the optimal approach with an iterative method such as a gradient algorithm[1] as in [9] and the suboptimal closed-form approach [10] based on a diagonal relaxation which ignores off-diagonal elements of $\mathbf{R}_\omega$, i.e., $\mathbf{\Phi} = \mathbf{\Phi}_d$ where $\mathbf{\Phi}_d \in \mathbb{C}^{N_t \times N_t}$ denotes a diagonal matrix. In this paper, the analytical performance of these two cooperative schemes [9] [10] will be studied. Since $\mathbf{\Phi}_d$ becomes globally optimal when $\mathbf{T} = \mathbf{0}$ or $N_t = 1$, our analysis includes conventional optimal designs for pure relaying channels without a direct link ($\mathbf{T} = \mathbf{0}$) [7] [8] and for the single stream cooperative transmission ($N_t = 1$) [14] as special cases.

## IV. Diversity-Multiplexing Tradeoff Analysis

Now, we evaluate the asymptotic high-SNR performance of MMSE-based cooperative relaying systems using the DMT analysis. We first propose an upperbound of the DMT which provides a theoretical limit of the system. Then, by showing that the upperbound is actually achievable by the optimal and the suboptimal schemes proposed in [9] and [10], respectively,

[1]This approach is equivalent to [9] where the optimal relay matrix $\hat{\mathbf{Q}}$ was directly found by a projected gradient method after the canonical coordination beamforming.

we specify the optimal DMT[2] for MMSE-based relaying systems. Several definitions and assumptions are given below.

Letting $R(\rho)$ and $P_e(\rho)$ denote the transmit rate and the error probability with the operating SNR $\rho$, respectively, the multiplexing gain $r$ and the corresponding diversity gain $d(r)$ are defined as [1]

$$\lim_{\rho\to\infty}\frac{R(\rho)}{\log\rho}=r \quad \text{and} \quad \lim_{\rho\to\infty}\frac{\log P_e(\rho)}{\log\rho}=-d(r),$$

and we write $P_e(\rho)\doteq\rho^{-d(r)}$ for notational simplicity. The inequalities $\dot{\leq}$ and $\dot{\geq}$ are similarly defined. Note that if the rate $R(\rho)$ is a constant over all SNR range, the multiplexing gain converges to zero. In this paper, the outage probability will be studied, since the outage performance of the mutual information (MI) gives a good approximation of the block error rate [1] [3]. Also, for simplicity of our analysis, we assume $P_T=P_R=N_t\rho$, but the result can be easily extended to more general cases.

Our main result is illustrated in the following theorem.

*Theorem 1:* For MMSE-based cooperative relaying systems with $N_t<\min(N_r,N_d)$, the optimal [9] and the suboptimal [10] relay matrices achieve the optimal DMT characterized as

$$d_{\text{MMSE}}(r)=(N_r+N_d-N_t+1)\left(1-\frac{2r}{N_t}\right)^+.$$

*Proof:*

*1) DMT Upperbound:* In half duplex relaying systems with the MMSE spatial equalizer and Gaussian input codeword jointly encoded across antennas[3], the MI can be defined as [11]

$$\mathcal{I}=\frac{1}{2}\sum_{k=1}^{N_t}\log\left(1+\gamma_k\right), \tag{4}$$

where $\gamma_k=\rho/[\mathbf{R}_e]_{kk}-1$. Then, using Jensen's inequality and eliminating the second term of $\mathbf{R}_e$ in (2), the MI is upperbounded by

$$\begin{aligned}\mathcal{I} &\leq \frac{N_t}{2}\log\left(\frac{1}{N_t}\sum_{k=1}^{N_t}\frac{\rho}{[\mathbf{R}_e]_{k,k}}\right)\\ &\leq \frac{N_t}{2}\log\left(\frac{1}{N_t}\sum_{k=1}^{N_t}\frac{1}{(\rho\mathbf{H}_T^H\mathbf{H}_T+\mathbf{I}_{N_t})^{-1}_{k,k}}\right),\end{aligned} \tag{5}$$

where $\mathbf{H}_T\triangleq[\mathbf{H}^T\ \mathbf{T}^T]^T\in\mathbb{C}^{(N_r+N_d)\times N_t}$. Now, we can check that the terms inside the logarithm in (5) exactly coincide with the point-to-point MIMO channel with $N_t$ transmit and $N_r+N_d$ receive antennas. Thus, following the previous result in [12], one can easily find an upperbound of the system DMT as

$$d_{\text{MMSE}}\leq(N_r+N_d-N_t+1)\left(1-\frac{2r}{N_t}\right)^+. \tag{6}$$

Note that the multiplexing gain $r$ multiplied by 2 in (6) is attributed to the half-duplex nature of the system, which means that $r\leq N_t/2$.

[2]The optimal DMT reveals the best possible error probability exponent $d(r)$ in the considered system configuration.

[3]This coding strategy is also called *vertical encoding* on which we focus here, but our result can be easily applied to *horizontal encoding* where data streams are separately encoded in each source antenna [11] [12].

*2) Achievability:* First, we characterize an MI lowerbound which describes the achievable DMT of existing relaying strategies [9] [10]. Since the function $-\log(\cdot)$ is convex, using the definition in (4) and Jensen's inequality again, we have

$$\begin{aligned}\mathcal{I} &\geq -\frac{N_t}{2}\log\left(\frac{1}{\rho N_t}\text{Tr}\big(\mathbf{R}_e\big)\right)\\ &= -\frac{N_t}{2}\log\left(\frac{1}{\rho N_t}\text{Tr}\big(\hat{\mathbf{\Phi}}^H\mathbf{\Lambda}_g\hat{\mathbf{\Phi}}+\mathbf{\Lambda}_\omega^{-1}\big)^{-1}+\sigma\right),\end{aligned}$$

where $\sigma\triangleq N_t^{-1}\text{Tr}\big(\rho\mathbf{H}_T^H\mathbf{H}_T+\mathbf{I}_{N_t}\big)^{-1}$. Since $\hat{\mathbf{\Phi}}$ is optimal [9] and the suboptimal solution $\mathbf{\Phi}_d$ [10] is also optimal under the diagonal structure, the setting $\hat{\mathbf{\Phi}}=\sqrt{\eta}\mathbf{I}_M$ clearly yields an MI lowerbound of both cases where $\eta$ can be chosen to be $\eta=P_R/(\rho N_t)$ from Lemma 2 (see Appendix) and the relay constraint in (3).

Then, by the assumption $P_R=N_t\rho$, we have $\eta=N_t/M\geq 1$ and it follows

$$\begin{aligned}\mathcal{I} &\geq -\frac{N_t}{2}\log\left(\frac{1}{\rho N_t}\text{Tr}\Big(\eta\mathbf{\Lambda}_g+\mathbf{\Lambda}_\omega^{-1}\Big)^{-1}+\sigma\right)\\ &\geq -\frac{N_t}{2}\log\Big(\frac{1}{N_t}\text{Tr}\Big(\rho\mathbf{\Lambda}_g+\rho\mathbf{\Lambda}_t+\mathbf{I}_M\Big)^{-1}+\sigma\Big),\end{aligned}$$

where the last inequality follows from Lemma 1 (see Appendix), because $\mathbf{A}\preceq\mathbf{B}$ implies $\text{Tr}(\mathbf{A}^{-1})\geq\text{Tr}(\mathbf{B}^{-1})$. The important feature to notice here is that diagonal elements of $\mathbf{\Lambda}_t$ are arranged in *ascending* order in contrast to $\mathbf{\Lambda}_g$.

Using this bound and setting the target data rate as $R(\rho)=r\log\rho$, we finally obtain the outage probability as

$$\begin{aligned}P_{\text{out}} &\triangleq P\left(\mathcal{I}\leq R(\rho)\right)\\ &\leq P\Big(\text{Tr}\big(\rho\mathbf{H}_T^H\mathbf{H}_T+\mathbf{I}_{N_t}\big)^{-1}\\ &\qquad +\text{Tr}\Big(\rho(\mathbf{\Lambda}_g+\mathbf{\Lambda}_t)+\mathbf{I}_M\Big)^{-1}\geq N_t\rho^{-\frac{2r}{N_t}}\Big)\\ &= P\left(\sum_{k=1}^{N_t}\left(\frac{1}{1+\rho\lambda_{ht,k}}+\frac{1}{1+\rho\lambda_{gt,k}}\right)\geq N_t\rho^{-\frac{2r}{N_t}}\right),\end{aligned} \tag{7}$$

where $\lambda_{ht,k}$ designates the $k$-th largest eigenvalue of $\mathbf{H}_T^H\mathbf{H}_T$ and $\lambda_{gt,k}\triangleq\lambda_{g,k}+\lambda_{t,k}$.

Now, we derive the outage probability exponent. In the exponential sense, the outage probability in (7) mainly depends on the worst case channel gain, and thus can be asymptotically upperbounded as

$$\begin{aligned}P_{\text{out}} &\dot{\leq} P\left(\frac{1}{\rho\lambda_{ht,N_t}}+\frac{1}{\min\limits_{k=1,\ldots,M}\{\rho\lambda_{gt,k}\}}\geq N_t\rho^{-\frac{2r}{N_t}}\right)\\ &\leq P\left(\Delta\leq\frac{2}{N_t}\rho^{-\left(1-\frac{2r}{N_t}\right)}\right),\end{aligned} \tag{8}$$

where $\Delta\triangleq\min(\lambda_{ht,N_t},\lambda_{gt,\min})$ and $\lambda_{gt,\min}\triangleq\min_{k=1,\ldots,M}\{\lambda_{gt,k}\}$. The last inequality follows from the harmonic mean bound $A^{-1}+B^{-1}\leq\frac{2}{\min(A,B)}$. We see that for the case of $2r/N_t\geq 1$, the outage exponent converges to zero. Hence, supposing $2r/N_t<1$, we find the near zero behavior of a distribution of $\Delta$ in the following.

For a small argument $\delta$, we recognize from Lemma 3 (see Appendix) that the cumulative distribution function (CDF) of

$\Delta$ is given by

$$F_\Delta(\delta) = F_{\lambda_{ht,N_t}}(\delta) + \sum_{k=1}^{M} F_{\lambda_{gt,k}}(\delta),$$

where $F_{\lambda_{ht,N_t}}(\cdot)$, and $F_{\lambda_{gt,k}}(\cdot)$ represent the CDFs of $\lambda_{ht,N_t}$ and $\lambda_{gt,k}$, respectively. It is also well known [15] that for an $m \times m$ complex Wishart matrix $\mathbf{S}^H\mathbf{S}$ with a Gaussian matrix $\mathbf{S} \in \mathbb{C}^{n\times m}$, its $k$-th largest (or smallest) eigenvalue $\lambda_k$ is polynomially distributed near zero as $F_{\lambda_k}(\lambda_k) \propto \lambda_k^{(m-k+1)^+(n-k+1)^+}$ (or $\propto \lambda_k^{k(n-m+k)^+}$).

Using these facts and employing Lemma 4 (see Appendix), we obtain $F_\Delta(\delta)$ as

$$\begin{aligned} F_\Delta(\delta) \quad &\propto \quad \delta^{(N_r+N_d-N_t+1)} \\ &\qquad + \sum_{k=1}^{M} \delta^{(N_d-k+1)(N_r-k+1) \ + \ k(N_d-N_t+k)} \\ &\simeq \quad \delta^{\min(D_h,D_g)}, \end{aligned}$$

where $D_h$ and $D_g$ are defined as

$$\begin{aligned} D_h &\triangleq N_r + N_d - N_t + 1 \\ D_g &\triangleq \min_{k=1,\ldots,N_t} \big((N_d-k+1)(N_r-k+1) \\ &\qquad + k(N_d - N_t + k)\big) \\ &= \min_{k=1,\ldots,N_t} \big(D_h + (N_r-k+1)(N_d-k) \\ &\qquad + (k-1)(N_d-N_t+k)\big). \end{aligned}$$

Then, we see that $D_g \geq D_h$ holds for all $k$, and thus the outage probability is readily acquired by

$$P\left(\mathcal{I} \leq R(\rho)\right) \dot{\leq} c\rho^{-D_h\left(1-\frac{2r}{N_t}\right)^+},$$

where $c$ is a constant and the resulting outage exponent lowerbound is $d_{\mathrm{MMSE}}(r) \geq (N_r+N_d-N_t+1)(1-2r/N_t)^+$, and the proof is completed. ■

Theorem 1 illustrates the best possible error probability exponent, i.e., the optimal DMT of MMSE-based cooperative relaying systems. From the result that the DMT upperbound in (6) is actually achievable, we recognize that the optimal DMT performance is determined by the broadcast phase in the first time slot. Another interesting insight here is that the DMT expression $(N_r + N_d - N_t + 1)(1 - 2r/N_t)^+$ coincides with that of a point-to-point open-loop MMSE [12] with $N_t$-transmit and $(N_r + N_d)$-receive antennas used at a rate $2R$. This implies that with a well-designed relay matrix, the relay and the destination act like a virtual single MMSE receiver. Theorem 1 also shows that the suboptimal approach based on the diagonal relaxation, i.e., $\hat{\mathbf{\Phi}} = \mathbf{\Phi}_d$ [10] incurs no performance loss in terms of DMT compared to the optimal iterative design [9]. Our result accounts for the analytical performance of conventional designs with $N_t = 1$ or $\mathbf{T} = \mathbf{0}$ as special cases. For example, the DMT for the single stream beamforming scheme with $N_t = 1$ [14] is readily acquired as $(N_r + N_d)(1-2r)^+$. Also, setting $\mathbf{T} = \mathbf{0}$ and following the proof of Theorem 1, we attain the DMT of the MMSE-based optimal pure relaying system [7] [8] as $(N_r - N_t + 1)(1 - 2r/N_t)^+$.

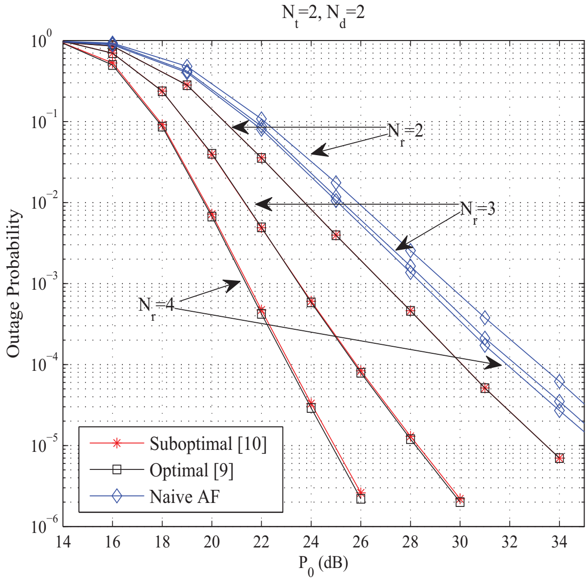


Fig. 2. Outage probability as a function of $P_0$ with $R = 5$ bpcu

## V. Numerical Results

In this section, we provide numerical results for the outage performance of various relaying strategies to illustrate our claims described in the previous section and demonstrate the accuracy of our analysis. We define $P_0$ as the total transmit power used at both the source and the relay for two time slots assuming $P_T = P_R = P_0/2$. The transmission rate $R$ is measured in bits per channel use (bpcu)[4].

Figures 2 presents simulations results for the outage performance of the optimal [9] and the suboptimal [10] relay matrix designs with $N_t = N_d = 2$ and various number of relay antennas at rate $R = 5$ bpcu. The Naive AF indicates the most simple scheme where only the power normalizing operation is performed at the relay, i.e., $\mathbf{Q} = \sqrt{P_R/\mathrm{Tr}(\rho\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})}\mathbf{I}_{N_r}$ which serves as a benchmark to validate our analysis. Note that with a fixed rate $R$, the multiplexing gain $r$ equals zero. In this configuration, the optimal DMT is expressed as $d_{\mathrm{MMSE}}(0) = N_r + 1$. We confirm from this figure that numerical performance of the optimal and the suboptimal relay matrices exactly behave as predicted by our analysis. On the contrary, we can check that the naive AF does not properly exploit the relay antennas especially when $N_r > 2$, which leads to a significant diversity loss.

Finally, Figure 3 presents the outage probability of $4\times4\times4$ systems with a non-zero multiplexing gain $r$, In other words, for each curve, the transmission rate is set to be a increasing function of SNR as $R(\rho) \doteq r \log \rho$ bpcu. Note that the DMT is given as $d_{\mathrm{MMSE}}(r) = 5(1-r/2)$ from Theorem 1. For all simulations in this figure, we confirm that our analysis accurately matches with the diversity order for various multiplexing gain $r$.

[4]We transmit $2R$ bits for two time slots.

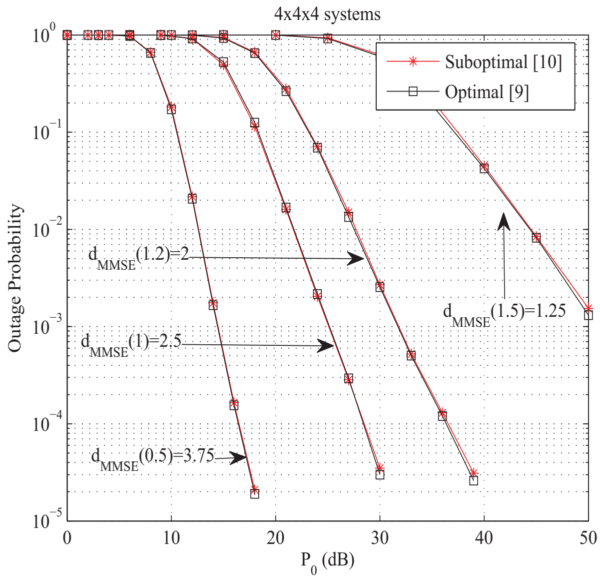


Fig. 3. Outage probability with non-zero multiplexing gain $r$

## VI. Conclusion

In this paper, we have studied the asymptotic high-SNR performance of MMSE-based cooperative MIMO half-duplex relaying systems using DMT analysis. First, we have established the DMT upperbound which offers theoretical limit of the system, and then its achievability has been proved. From our analysis, it is shown that the suboptimal closed-form design as well as the optimal iterative design for the relay amplifying matrix achieve the optimal DMT which has been evaluated in our analysis. Several interesting observations have also been made. Finally, through numerical simulations, we have verified the accuracy of our analysis.

## Appendix

In this appendix, we introduce several useful lemmas which have been exploited in our analysis.

*Lemma 1:* Define a diagonal matrix $\mathbf{\Lambda}_t \in \mathbb{C}^{N_t \times N_t}$ whose diagonal entries consist of the eigenvalues of $\mathbf{T}^H\mathbf{T}$ arranged in *ascending* order, i.e., $\lambda_{t,1} \leq, \lambda_{t,2} \leq, \dots, \leq \lambda_{t,N_t}$. Then, we have $\mathbf{\Lambda}_\omega^{-1} \succ \mathbf{\Lambda}_t + \rho^{-1}\mathbf{I}_{N_t}$ where $\succ$ (or $\prec$) represents the generalized inequality defined on the positive definite cone.

*Proof:* After some mathematical manipulations, $\mathbf{\Omega}$ in (2) can be modified as $\mathbf{\Omega} = \mathbf{R}_T - \left(\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1}$. Since $\mathbf{A} = \mathbf{B} - \mathbf{C}$ implies $\mathbf{A} \prec \mathbf{B}$ for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{S}_{++}^{N_t}$, it must be true that $\mathbf{\Omega} \prec \mathbf{R}_T$. In other words, assuming that the eigenvalues of $\mathbf{R}_T$ are arranged in descending order, we have $\mathbf{\Lambda}_\omega \prec (\mathbf{\Lambda}_t + \rho^{-1}\mathbf{I}_{N_t})^{-1}$ and conversely we obtain $\mathbf{\Lambda}_\omega^{-1} \succ \mathbf{\Lambda}_t + \rho^{-1}\mathbf{I}_{N_t}$, and the lemma is proved. ■

*Lemma 2:* At high SNR, $\mathbf{R}_\omega$ defined in problem (3) is upperbounded by $\mathbf{R}_\omega \prec \rho\mathbf{I}_{N_t}$.

*Proof:* $\mathbf{R}_\omega$ can be approximated at high SNR as

$$\begin{aligned} \mathbf{R}_\omega &= \rho\mathbf{U}_\omega^H\mathbf{L}(\mathbf{H}^H\mathbf{H} + \rho^{-1}\mathbf{I}_{N_t})\mathbf{L}^H\mathbf{U}_\omega \\ &\approx \rho\mathbf{U}_\omega^H(\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1})^{-1}(\mathbf{H}^H\mathbf{H})^2 \\ &\quad \times (\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1})^{-1}\mathbf{U}_\omega. \end{aligned}$$

Since we have $\mathbf{H}^H\mathbf{H} \prec \mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1}$, it is immediate that $\mathbf{R}_\omega \prec \rho\mathbf{I}_M$. ■

*Lemma 3 ( [16]):* For $K$ positive random variables $\{X_i\}_{i=1,\dots,K}$[5], define $W \triangleq \min(X_1, X_2, \dots, X_K)$. Then, for a small argument $w$ (i.e., $w \to 0^+$), the CDF of $W$ is given by $F_W(w) = F_{X_1}(w) + F_{X_2}(w) + \cdots + F_{X_K}(w)$ where $F_{X_i}(\cdot)$ indicates the CDF of $X_i$.

*Lemma 4:* Let us define two independent and polynomially distributed random variables $Y \geq 0$ and $Z \geq 0$, i.e., $F_Y(y) = c_1y^\alpha$ and $F_Z(z) = c_2z^\beta$ where $\alpha \geq 1$ and $\beta \geq 1$, and $c_1$ and $c_2$ are constants. Then, the CDF of $S = Y + Z$ equals $F_S(s) = \kappa s^{\alpha+\beta}$ where $\kappa = \frac{c_1c_2\alpha!\beta!}{(\alpha+\beta)!}$.

*Proof:* The CDF of $S$ can be written in a convolution form as $F_S(s) = \int_0^s \alpha c_1 c_2 y^{\alpha-1}(s-y)^\beta dy$. By solving the integral, we simply obtain the result. Details are trivial, and thus skipped. ■

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[5] Here, the random variables $X_1, \dots, X_K$ do not need to be independent.