The Equivalence of Two Unified Solutions for Optimum Space–Time Processing

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***Abstract*—This letter presents a simple proof of the identity of two unified solutions for optimum space–time processing given in a previous paper.**

***Index Terms*—Equalization, interference suppression, space–time processing.**

I. Introduction

In [1], we provided a unified analysis of optimum space–time processors based on the following two analytical diversity receiver models: 1) a general model—with a linear filter on each diversity branch as shown in Fig. 1(a) and 2) a "matched filter" model—with a bank of matched filters on each branch, followed by common filters, as shown in Fig. 1(b). Closed-form results were given for each receiver model, in terms of the minimum mean-square-error or maximum signal-to-noise ratio solutions for different types of equalizers, including a linear equalizer (LE), a decision-feedback equalizer (DFE), and a maximum-likelihood sequence estimator (MLSE). Although we implied in [1] that the two receiver models lead to the same optimum solutions, this requires some proof that is not directly obtainable from the results we presented. We therefore provide such a proof here for the completeness of the overall unified analysis.

II. Two Closed-Form Solutions

We first rewrite (mostly repeat) the results of [1] and define all the variables and notation. All the expressions are given in the frequency-domain. In Fig. 1(a) and (b), $R_j(f)$ is the Fourier transform of the received signal at the jth diversity antenna, M is the total number of antennas, and L is the number of cochannel interference (CCI) sources. Note that these figures only show the diversity combining part of the receiver. A complete space–time equalizer receiver would include subsequent elements, such as a slicer and feedback filter for DFE and a Viterbi processor for MLSE (see [1] for the detailed structures).

A. General Model

The diversity receiver in Fig. 1(a) consists of a linear filter $W_j(f)$, $j = 0, 1, \cdots, M-1$, on each branch. The combiner output is

$$Z(f) = \sum_{j=0}^{M-1} \sum_{m=-J}^{J} W_j\left(f - \frac{m}{T}\right) R_j\left(f - \frac{m}{T}\right) \tag{1}$$

where the summation with respect to m is a result of spectrum folding due to symbol-rate sampling (the limit of this summation is finite because the desired and CCI sources are assumed to be band-limited). The optimum filter solution was given in [1] in the form of

$$\boldsymbol{W} = \boldsymbol{R}^{-1}\boldsymbol{H}_0^* S(f) = \boldsymbol{R}_{I+N}^{-1}\boldsymbol{H}_0^* \frac{S(f)}{1+\Gamma(f)} \tag{2}$$

where

$$\boldsymbol{W} \triangleq \left[W_0\left(f - \frac{J}{T}\right) \cdots W_{M-1}\left(f - \frac{J}{T}\right) \cdots \cdots W_0\left(f + \frac{J}{T}\right) \cdots W_{M-1}\left(f + \frac{J}{T}\right)\right]^T \tag{3}$$

$$\boldsymbol{H}_i \triangleq \left[H_{i0}\left(f - \frac{J}{T}\right) \cdots H_{i,M-1}\left(f - \frac{J}{T}\right) \cdots \cdots H_{i0}\left(f + \frac{J}{T}\right) \cdots H_{i,M-1}\left(f + \frac{J}{T}\right)\right]^T \tag{4}$$

$$\boldsymbol{R} \triangleq \sum_{i=0}^{L} \boldsymbol{H}_i^* \boldsymbol{H}_i^T + N_0 \boldsymbol{I} \tag{5}$$

$$\boldsymbol{R}_{I+N} \triangleq \sum_{i=1}^{L} \boldsymbol{H}_i^* \boldsymbol{H}_i^T + N_0 \boldsymbol{I} \tag{6}$$

$$\Gamma(f) = \boldsymbol{H}_0^T \boldsymbol{R}_{I+N}^{-1} \boldsymbol{H}_0^*. \tag{7}$$

$H_{ij}(f)$ is the frequency response of the channel corresponding to the signal source i on diversity branch j, $\boldsymbol{R}$ and $\boldsymbol{R}_{I+N}$ are the correlation matrices of the overall received signal and the interference plus noise, respectively, N_0 is the two-sided noise power density at each antenna, $\boldsymbol{I}$ is the identity matrix, $\Gamma(f)$ is the signal-to-interference-plus-noise power density ratio, and the superscripts * and T denote complex

Paper approved by K. B. Letaief, the Editor for Wireless Systems of the IEEE Communications Society. Manuscript received May 12, 1999; revised August 25, 1999.

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Publisher Item Identifier S 0090-6778(00)02267-4.

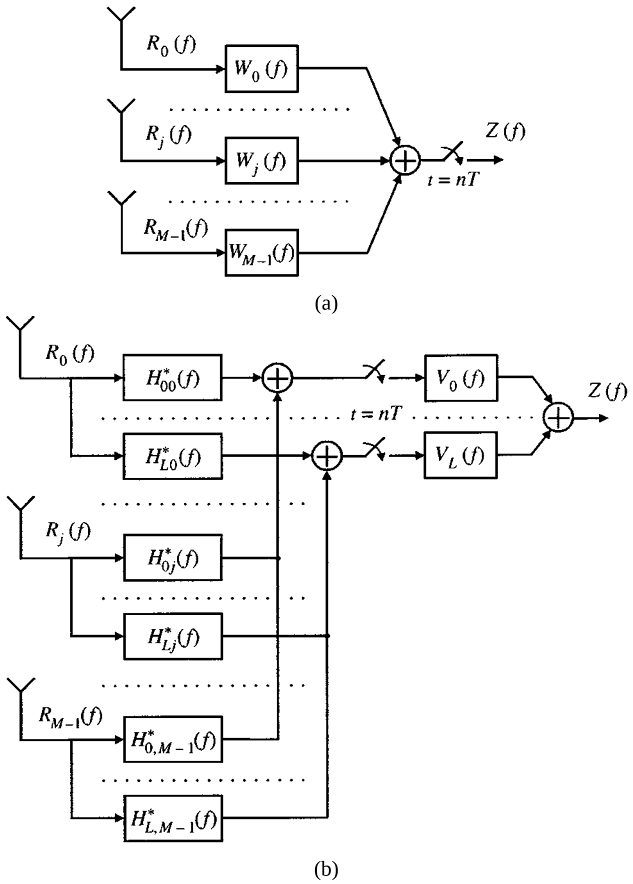

Fig. 1. Two diversity receiver models: (a) the general model and (b) the "matched filter" model.

conjugate and transpose, respectively. In (2), $S(f)$ is a temporal filter which can be given as

$$S(f) = \begin{cases} 1, & \text{for LE} \\ C[1+\Gamma(f)], & \text{for DFE} \\ (1+\Gamma(f))\dfrac{C[\Gamma(f)]}{\Gamma(f)}, & \text{for MLSE} \end{cases} \tag{8}$$

where $C[\cdot]$ denotes *canonical factor* as defined in [1].

B. "Matched Filter" Model

The receiver in Fig. 1(b) assumes the use of a bank of matched filters $\{H_{ij}^*(f)\}$ which, after diversity combining, is followed by a bank of symbol-spaced transversal filters $\{V_i(f)\}$, each corresponding to the signal source i. The optimum solution for $\boldsymbol{V} = [V_0(f) \cdots V_L(f)]^T$ is given in [1] as

$$\boldsymbol{V} = (\boldsymbol{P} + N_0\boldsymbol{I})^{-1}\boldsymbol{U}S(f) \tag{9}$$

where

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{H}_0^T \\ \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} [\boldsymbol{H}_0^*\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] \tag{10}$$

$U = [1, 0, \cdots, 0]^T$ is a column vector with $L+1$ rows, and $S(f)$ has the same expression as (8), except that $\Gamma(f)$ is now given by

$$\Gamma(f) = \frac{1}{N_0}\frac{1}{\boldsymbol{U}^T(\boldsymbol{P}+N_0\boldsymbol{I})^{-1}\boldsymbol{U}} - 1. \tag{11}$$

III. Proof of Equivalence

We now prove that the receiver models in Fig. 1(a) and (b) lead to an identical optimum solution. Note in Fig. 1(b) that the combined response of the matched filters and common filters can be given as

$$[\boldsymbol{H}_0^*\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*]\boldsymbol{V} = \sum_{i=0}^{L} \boldsymbol{H}_i^* V_i(f). \tag{12}$$

Thus, our proof is to show that the optimum $\boldsymbol{W}$ in (2) is identical to (12) with $\boldsymbol{V}$ given by (9). We divide this into three steps.

Step 1: We first prove that $S(f)$ in (2) is identical to $S(f)$ in (9). Since they are both given by (8), except for the different expressions for $\Gamma(f)$, the proof is simply to show that these expressions, (7) and (11), are identical.

Using the Schur complement [2, Appendix][1] and the expression for $\boldsymbol{P}$ in (10), we can write (13), shown at the bottom of the page. Therefore, (11) becomes

$$\Gamma(f) = \frac{\boldsymbol{H}_0^T\boldsymbol{H}_0^*}{N_0} - \frac{1}{N_0}\boldsymbol{H}_0^T[\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] \cdot \left[\begin{bmatrix} \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix}[\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] + N_0\boldsymbol{I}\right]^{-1} \begin{bmatrix} \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} \boldsymbol{H}_0^*. \tag{14}$$

On the other hand, we can rewrite (7) as

$$\Gamma(f) = \boldsymbol{H}_0^T \left[[\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*]\begin{bmatrix} \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} + N_0\boldsymbol{I}\right]^{-1} \boldsymbol{H}_0^*. \tag{15}$$

The identity of (14) and (15) can be shown using the matrix inversion lemma [3, Appendix D].

[1]The Schur complement Δ of the submatrix $\boldsymbol{D}$ of a matrix $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$ is $\Delta = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$. The upper-left submatrix of $\boldsymbol{X}^{-1}$ is Δ^{-1}.

$$\boldsymbol{U}^T(\boldsymbol{P}+N_0\boldsymbol{I})^{-1}\boldsymbol{U} = \frac{1}{\boldsymbol{H}_0^T\boldsymbol{H}_0^* + N_0 - \boldsymbol{H}_0^T[\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*]\left[\begin{bmatrix} \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix}[\boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] + N_0\boldsymbol{I}\right]^{-1}\begin{bmatrix} \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix}\boldsymbol{H}_0^*} \tag{13}$$

Step 2: We show that $\boldsymbol{W}$ can be rewritten as

$$\boldsymbol{W} = [\boldsymbol{H}_0^* \boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*]\boldsymbol{V}' = \sum_{i=0}^{L} \boldsymbol{H}_0^* V_2'(f) \qquad (16)$$

where

$$\begin{aligned} \boldsymbol{V}' &= [V_0'(f) \cdots V_L'(f)]^T \\ V_0'(f) &= \frac{S(f) - \boldsymbol{H}_0^T \boldsymbol{W}}{N_0} \\ V_i'(f) &= -\frac{\boldsymbol{H}_i^T \boldsymbol{W}}{N_0}, \qquad i = 1, \cdots, L. \end{aligned} \qquad (17)$$

Proof:

$$\sum_{i=0}^{L} \boldsymbol{H}_0^* V_i'(f) \overset{(16)}{=} \boldsymbol{H}_0^* \frac{S(f) - \boldsymbol{H}_0^T \boldsymbol{W}}{N_0} - \sum_{i=1}^{L} \boldsymbol{H}_i^* \frac{\boldsymbol{H}_i^T \boldsymbol{W}}{N_0}.$$

$\square$

$$\begin{aligned} &= \frac{\boldsymbol{R}\boldsymbol{R}^{-1} \boldsymbol{H}_0^* S(f) - \sum_{i=0}^{L} \boldsymbol{H}_i^* \boldsymbol{H}_i^T \boldsymbol{W}}{N_0} \\ &\overset{(2)}{=} \frac{\boldsymbol{R}\boldsymbol{W} - \sum_{i=0}^{L} \boldsymbol{H}_i^* \boldsymbol{H}_i^T \boldsymbol{W}}{N_0} \\ &= \frac{\left(\boldsymbol{R} - \sum_{i=0}^{L} \boldsymbol{H}_i^* \boldsymbol{H}_i^T\right) \boldsymbol{W}}{N_0} \\ &\overset{(5)}{=} \frac{N_0 \boldsymbol{I}}{N_0} \boldsymbol{W} = \boldsymbol{W}. \end{aligned}$$

Step 3: Finally, we prove that (12) and (16) are identical by showing that $\boldsymbol{V}$ in (9) is identical to $\boldsymbol{V}'$ in (17). From (9) and (10), we have

$$\boldsymbol{V} = \left(\begin{bmatrix} \boldsymbol{H}_0^T \\ \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} [\boldsymbol{H}_0^* \boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] + N_0 \boldsymbol{I} \right)^{-1} \boldsymbol{U} S(f). \qquad (18)$$

Using the matrix inversion lemma [3, Appendix D], we can rewrite the above equation as

$$\boldsymbol{V} = \left(\frac{\boldsymbol{I}}{N_0} - \frac{1}{N_0} \begin{bmatrix} \boldsymbol{H}_0^T \\ \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} \left[[\boldsymbol{H}_0^* \boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] \begin{bmatrix} \boldsymbol{H}_0^T \\ \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} + N_0 \boldsymbol{I} \right]^{-1} \cdot [\boldsymbol{H}_0^* \boldsymbol{H}_1^* \cdots \boldsymbol{H}_L^*] \right) \boldsymbol{U} S(f). \qquad (19)$$

Using (5) and (2), we obtain

$$\begin{aligned} \boldsymbol{V} &= \begin{bmatrix} \frac{S(f)}{N_0} \\ 0 \\ \cdots \\ 0 \end{bmatrix} - \frac{1}{N_0} \begin{bmatrix} \boldsymbol{H}_0^T \\ \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} \boldsymbol{R}^{-1} \boldsymbol{H}_0^* S(f) && (20) \\ &= \begin{bmatrix} \frac{S(f)}{N_0} \\ 0 \\ \cdots \\ 0 \end{bmatrix} - \frac{1}{N_0} \begin{bmatrix} \boldsymbol{H}_0^T \\ \boldsymbol{H}_1^T \\ \cdots \\ \boldsymbol{H}_L^T \end{bmatrix} \boldsymbol{W} && (21) \end{aligned}$$

which is identical to $\boldsymbol{V}'$ in (17). Our proof is therefore complete.

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