Quasi-Cyclic Low-Density Parity-Check Codes for Space-Time Bit-Interleaved Coded Modulation

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Abstract—In this letter, it is shown that the diversity order of space-time bit-interleaved coded modulation (ST-BICM) system is determined by the number of submatrices having linearly independent column vectors in a parity-check matrix of quasi-cyclic low-density parity-check (QC-LDPC) code. It is also proved that this diversity order can be derived from the base matrix of QC-LDPC code, which can make it easy to design QC-LDPC codes suitable for ST-BICM systems. Finally, the simulation results are provided to confirm the analytical results.

Index Terms—Quasi-cyclic low-density parity-check (QC-LDPC) codes, space-time bit-interleaved coded modulation (ST-BICM), diversity order.

I. INTRODUCTION

LOW-density parity-check (LDPC) codes have a remarkable capacity-approaching performance with iterative decoding over additive white Gaussian noise (AWGN) channel. Also, LDPC codes have advantages such as parallel decoding and self error-correction capability using syndrome check. In the case of random LDPC codes, a significant amount of memory is needed to store parity-check matrices and it is hard to access the memory and encode/decode data efficiently. Therefore, structured LDPC codes constructed by algebraic methods are needed for the implementation purpose.

Quasi-cyclic LDPC (QC-LDPC) codes [1] can solve the memory problem due to their linear-time encodability and small size of required memory. They can be encoded using simple shift-registers and decoded efficiently. Compared to random LDPC codes of the same length, the required memory size for storing the parity-check matrices of QC-LDPC codes can be reduced by a factor of z, when $z \times z$ circulant permutation matrices (circulants) or zero matrix are employed.

Coded modulation schemes are designed to obtain time diversity on antenna radio links and bit-interleaved coded modulation (BICM) utilizes the code diversity using iterative decoding between a demapper and a decoder [2], [3]. Furthermore, BICM is extended to MIMO systems to achieve higher diversity gains, which is called space-time BICM (ST-BICM) [4], [5], [6]. In case of block fading channels, it was shown that the maximum diversity gain for ST-BICM systems

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is achieved if outgoing signals from every transmit antenna have a non-zero minimum Hamming distance for any pair of coded bit sequences [4], by deriving the average pairwise error probability of the maximum likelihood (ML) decoder.

In this letter, on the narrowband block fading channel [7], the diversity order of ST-BICM systems using QC-LDPC code is analyzed and it is shown that the diversity order can be derived from the base matrix of QC-LDPC code. The simulation results are given to confirm the analytical results.

II. CONSTRUCTION OF QC-LDPC CODES FOR ST-BICM

Let N_t and N_r denote the numbers of transmit and receive antennas in ST-BICM systems, respectively. Also, we assume the narrowband block fading channel [7] where the fading coefficients are quasi-static over a block of transmitted symbols and independent over blocks. Then, the diversity order of ST-BICM system is equal to the number of transmit antennas sending subcodewords with nonzero Hamming distance [4]. Our research will focus on deriving the number of subcodewords with nonzero Hamming distance in QC-LDPC codes.

A QC-LDPC code can be represented by a parity-check matrix consisting of zero matrices and circulants [1]. Let P^i be the circulant obtained by column shifting the $z \times z$ identity matrix to the right i times and P^∞ denote the $z \times z$ zero matrix. Then, an $m \times n$ parity-check matrix is defined by

$$H = \begin{bmatrix} P^{a_{11}} & P^{a_{12}} & \cdots & P^{a_{1n_s}} \\ P^{a_{21}} & P^{a_{22}} & \cdots & P^{a_{2n_s}} \\ \vdots & \vdots & \vdots & \vdots \\ P^{a_{m_s1}} & P^{a_{m_s2}} & \cdots & P^{a_{m_sn_s}} \end{bmatrix} \quad (1)$$

where $a_{ij} \in \{0, 1, \cdots, z-1, \infty\}$, $m = zm_s$, and $n = zn_s$. Also, a parity-check matrix can be represented by the shift values of circulants, called the $m_s \times n_s$ *shift matrix* H_s, as

$$H_s = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n_s} \\ a_{21} & a_{22} & \cdots & a_{2n_s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m_s1} & a_{m_s2} & \cdots & a_{m_sn_s} \end{bmatrix} . \quad (2)$$

H can be obtained by quasi-cyclic extension of H_s using $z \times z$ circulants and all-zero matrices, which will be denoted by $H = E(H_s, z)$. Unless mentioned otherwise, the sizes of H and H_s are $m \times n$ and $m_s \times n_s$, respectively. Also, we define a *base matrix* $H_b = \theta(H_s)$ where the function θ replaces the finite shift value and ∞ in H_s by 1 and 0, respectively.

To utilize the rich structure of QC-LDPC codes, we assume that N_t divides n_s and the symbols in the same circulant are

transmitted through the same transmit antenna. Consider the QC-LDPC code with the following parity-check matrix.

$$H(\{1,2,\cdots,N_t\}) = \begin{bmatrix} H_1 & H_2 & H_3 & \cdots & H_{N_t} \end{bmatrix} \quad (3)$$

where H_i is the ith $m \times \eta$ submatrix of H with $\eta = n/N_t$. For $A \subset \{1,2,\cdots,n_t\}$, we define an $m \times |A|\eta$ matrix $H(A)$ by taking H_i's, for $i \in A$ where $|A|$ denotes the size of A. Let $\underline{c} = [\underline{c}_1\underline{c}_2\cdots\underline{c}_{N_t}]$ be a codeword where $\underline{c}_i = (c_{i,0}, c_{i,1}, \cdots, c_{i,\eta-1})$. Then, the ith subcodeword $\underline{c}_i$ is transmitted at the ith transmit antenna in ST-BICM systems.

Next, we will derive lemma and theorem necessary for constructing QC-LDPC codes suitable for ST-BICM systems.

Lemma 1: For a ST-BICM system with N_t transmit and N_r receive antennas using QC-LDPC code with the parity-check matrix in (3), the diversity order is $(D+1)N_r$ if and only if for any subset $A \subset \{1,2,\cdots,N_t\}$ with $|A| \leq D$, the column vectors of $H(A)$ are linearly independent.

Proof: For simplicity, we assume that $N_r = 1$. Suppose that the diversity order is $d < D+1$. Then, there should be two codewords $\underline{\alpha}$ and $\underline{\beta}$ with zero Hamming distance at the subcodeword i, $i \in \{i_1, i_2, \cdots, i_{N_t-d}\}$. Since QC-LDPC code is a linear code, $\underline{\alpha} + \underline{\beta} = \underline{\gamma}$ is also a codeword and the subcodeword i of $\underline{\gamma}$ is $\underline{0}$ for $i \in \{i_1, i_2, \cdots, i_{N_t-d}\}$. Let A be $\{1,2,\cdots,N_t\}/\{i_1, i_2, \cdots, i_{N_t-d}\}$. Then, there are linearly dependent column vectors in $H(A)$, which is a contradiction.

For the converse, suppose that the column vectors of $H(A)$ with $A = \{i_1, i_2, \cdots, i_D\}$ are linearly dependent. Then, there exists a codeword with 1's in the positions corresponding to the linearly dependent column vectors of $H(A)$ and 0's in other positions. This codeword can be represented as $\underline{c} = [\underline{c}_1\underline{c}_2\cdots\underline{c}_{N_t}]$ with $\underline{c}_i = \underline{0}$ if $i \notin A$. Since this codeword and an all-zero codeword have nonzero Hamming distance only at the subcodeword i for $i \in A$, the diversity order is less than or equal to $|A| = D$, which is a contradiction. $\square$

For a given QC-LDPC code, the column permutation in H generates different submatrices H_i in (3) and changes the diversity order. Therefore, the column permutation should be carefully performed to maximize the diversity order.

Corollary 1: For a ST-BICM system with N_t transmit and N_r receive antennas using QC-LDPC code of rate R_c, the maximum achievable diversity is

$$\left(1 + \left\lfloor N_t(1 - R_c) \right\rfloor\right)N_r \quad (4)$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. $\square$

Lemma 1 and Corollary 1 are valid for any linear code. Note that (4) is the Singleton bound which was given in [7].

Theorem 1: For any linearly independent column vectors in $\theta(H_s)$, the corresponding quasi-cyclic extended column vectors in $H = E(H_s, z)$ are also linearly independent.

Proof: Let $\theta(H_s) = [h_1 h_2 \cdots h_{n_s}]$ where h_i is the $m_s \times 1$ column vector and $H = E(H_s, z) = [H_1 H_2 \cdots H_{n_s}]$ where H_i is the $m_s z \times n_s z$ submatrix obtained from the quasi-cyclic extension of h_i. Suppose that the columns of $H_{i_1}, H_{i_2}, \ldots,$ and H_{i_l} are linearly dependent, i.e., we have

$$\sum_{j=1}^{z}(H_{i_1})_j + \sum_{j=1}^{z}(H_{i_2})_j + \cdots + \sum_{j=1}^{z}(H_{i_l})_j = \underline{0} \bmod 2. \quad (5)$$

Since the columns of any circulant are added to the $z \times 1$ all-one column vector, it is clear that (5) is equivalent to $h_1 + h_2 + \cdots + h_{n_s} = \underline{0} \bmod 2$, i.e., $h_1, h_2,\ldots, h_{n_s}$ are linearly dependent. $\square$

From Lemma 1 and Theorem 1, we can see that the diversity order of ST-BICM systems is maintained after quasi-cyclic extension if N_t divides n_s. Therefore, the diversity order can be obtained from the base matrix, which makes it easy to construct QC-LDPC code suitable for ST-BICM systems.

III. NUMERICAL RESULTS

For simulation, BPSK modulation with 2 or 3 transmit and 1 receive antennas are assumed for block fading channels. To decode QC-LDPC codes, sum-product algorithm and 10 iterations are used.

We first consider the ST-BICM system with $N_t = 2$. The following simple base matrix is constructed for simulation.

$$H_{b_1}(\{1,2\}) = \begin{bmatrix} I_{12} & D_{12} \end{bmatrix},$$

where I_{12} is the 12×12 identity matrix and the $k \times k$ matrix D_k is defined as follows.

$$D_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since it is obvious that $H_{b_1}(\{1\}) = I_{12}$ and $H_{b_1}(\{2\}) = D_{12}$ are invertible matrices, the diversity order of the ST-BICM system using $H_{b_1}(\{1,2\})$ is 2 by Theorem 1. To show the effect of column permutations, $H_{b_2}(\{1,2\})$ is obtained by exchanging the 12th column and the 13th column in $H_{b_1}(\{1,2\})$. Since the 1st column and the 12th column of $H_{b_2}(\{2\})$ are identical, $H_{b_2}(\{2\})$ is not an invertible matrix and the diversity order of the ST-BICM system using $H_{b_2}(\{1,2\})$ is 1. Also, we use the QC-LDPC code of rate $1/2$ adopted in IEEE 802.16e standard [8], of which the shift matrix is given as follows.

$$H_b = [H_I | H_P],$$

where

$$H_I = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and H_P is the dual diagonal part given in [8]. It is clear that the diversity order is 1 for any column permutation since the 6th, 8th, and 12th columns in H_I are identical. These analytical results are confirmed by the simulation result in Fig. 1.
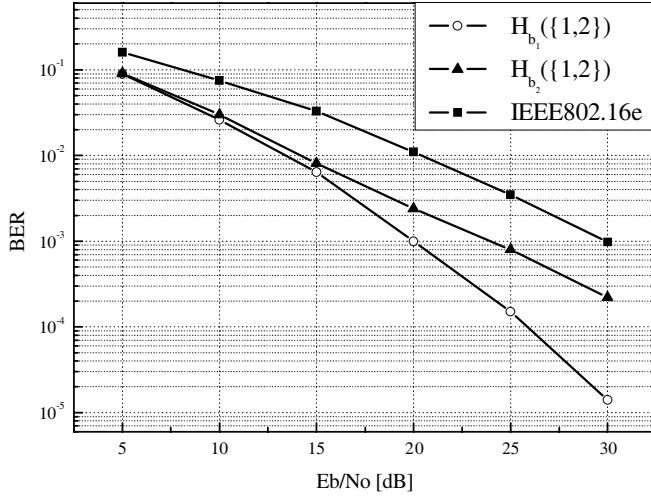
Fig. 1. Comparison of diversity orders of ST-BICM systems with various QC-LDPC codes when $N_t = 2$.
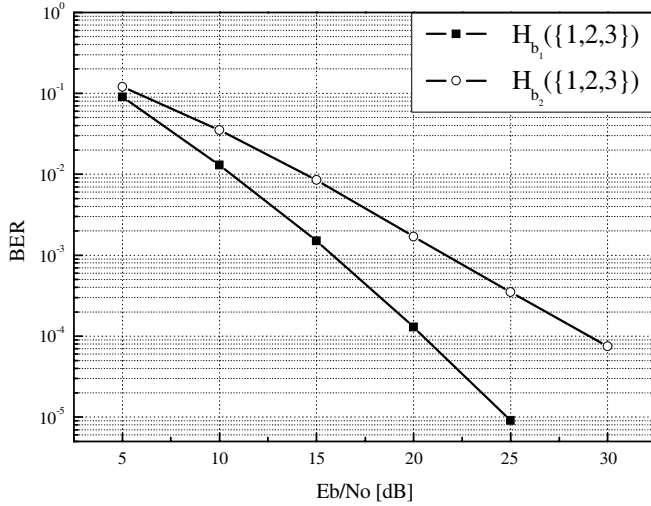

Fig. 2. Comparison of diversity orders of ST-BICM system with various QC-LDPC codes when $N_t = 3$.

Next, we compare the ST-BICM systems with $N_t = 3$. The following simple base matrices of QC-LDPC codes of rate $1/3$ are constructed for simulation.

$$H_{b_1}(\{1,2,3\}) = \begin{bmatrix} \alpha_7 & D_7 & 0_7 \\ \beta_7 & 0_7 & D_7 \end{bmatrix},$$

$$H_{b_2}(\{1,2,3\}) = \begin{bmatrix} \gamma_7 & D_7 & 0_7 \\ \delta_7 & 0_7 & D_7 \end{bmatrix},$$

where

$$\alpha_7 = \begin{bmatrix} I(1) & I(3) & I(4) & I(2) & I(6) & I(7) & I(5) \end{bmatrix},$$

$$\beta_7 = \begin{bmatrix} I(4) & I(2) & I(1) & I(3) & I(7) & I(6) & I(5) \end{bmatrix},$$

$I(i)$ denotes the ith column of 7×7 identity matrix,

$$\gamma_7 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\delta_7 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and 0_7 denotes the 7×7 all-zero matrix.

α_7 and β_7 are invertible matrices because they are permutation matrices and γ_7 and δ_7 are not invertible because the sum of all column vectors of each matrix is 0 under modulo-2 addition. Since $H_{b_1}(\{1,3\})$, $H_{b_1}(\{1,2\})$, and $H_{b_1}(\{2,3\})$ are invertible matrices, the diversity order of the ST-BICM system using $H_{b_1}(\{1,2,3\})$ is 3. Since the sum of all column vectors of $H_{b_2}(\{1\})$ is zero, the diversity order of the ST-BICM system using $H_{b_2}(\{1,2,3\})$ is 1. These analytical results are confirmed by the simulation result in Fig. 2.

Note that our simulation results also show the similar trend for other channel configurations such as channels with antenna correlation. In this letter, the theoretical diversity is derived for the ML decoding, while the simulation is performed using the suboptimal algorithm (i.e. sum-product algorithm). It is well known that the performance of this suboptimal algorithm is very close to that of the ML decoding and these two algorithms show the similar performance trend. Therefore, the simulation results can be used to confirm the theoretical results.

For future work, it will be interesting to find an efficient algebraic construction method of base matrices for QC-LDPC codes to achieve the maximum diversity order.

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