Asymptotic Ergodic Capacity Analysis for MIMO Amplify-and-Forward Relay Networks

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***Abstract*—In this letter, we analyze asymptotic ergodic capacity of MIMO amplify-and-forward (AF) relaying systems which employ linear processing at the relay. By exploiting the asymptotic results for eigenvalue distributions, we derive the ergodic capacity in various asymptotic antenna regimes as a closed-form expression with arbitrary system parameters. The analyzed results demonstrate that increasing the number of source antennas causes the capacity shrink phenomenon which is analogous to the channel hardening effect in multi-user MIMO systems. Although we assume asymptotically large antennas to obtain the closed-form expressions, simulation results show that our derived expressions are surprisingly accurate even with the moderate number of antennas, and thus can serve for analyzing practical MIMO relay networks.**

***Index Terms*—Multiple antennas, relay, ergodic capacity.**

I. INTRODUCTION

SINCE wireless relaying transmission is able to extend coverage and improve system performance, relay networks have been intensively studied [1][2]. In order to enhance the capacity of the relay network, multiple antennas can be considered for obtaining a similar performance benefit observed in point-to-point multiple-input multiple-output (MIMO) systems [3][4][5]. The capacity of MIMO relay systems has also been analyzed in [3] and [4], but still remains as open problems except some special cases.

It was recently shown in [6] and [7] that a linear technique based on singular value decomposition (SVD) achieves the capacity of MIMO amplify-and-forward (AF) relaying systems when the channel state information (CSI) is available at the relay. In practical relay networks, an AF method shows advantages of simple implementation and low computational complexity compared to decode-and-forward (DF) systems, since the relay node linearly processes only the received baseband signal without decoding the information. Therefore, the capacity analysis of AF MIMO relay systems is important for designing practical relay systems. In [8], the ergodic capacity of "naive" MIMO AF relay channels without the CSI at the relay node is analytically studied, where an identity matrix is employed as the relay filter matrix. However, the performance of such a naive AF scheme is noticeably inferior with respect to the SVD-based scheme which exploits the CSI.

In this letter, we study the ergodic capacity of MIMO AF systems with the CSI at both relay and destination nodes, where the capacity optimal SVD-based relay method [6][7] is employed. Computation of the exact ergodic capacity is extremely complicated, since the expectation operation of the instantaneous capacity should be carried out over the channel distribution. Moreover, this provides no helpful insights due to the absence of a closed-form solution. Hence, in this letter, by exploiting the Tracy-Widom law [9] for the eigenvalue distribution of a complex Wishart matrix in various asymptotic regimes, we derive new asymptotic closed-form expressions on the ergodic capacity of MIMO AF systems to provide insightful observations. We consider three asymptotic analyses: i) large relay, and fixed source and destination antennas, ii) large source and relay, and fixed destination antennas, and iii) large relay and destination, and fixed source antennas. Although this paper considers relaying systems with a single node, the analyzed results can be used as an upper bound of the ergodic capacity of cooperative relay networks clustered with multiple nodes in [10].

From the derived closed-form ergodic capacity, we learn that the capacity with large relay antennas is degraded if the number of source antennas becomes larger than that of destination antennas. This phenomenon resembles the channel hardening effect observed in multi-user diversity systems [11], which implies that from information theoretic view, it is not useful to increase source antennas more than the rank of the total effective channel. Also, our analysis demonstrates that the optimal power distribution between source and relay nodes is no longer uniform if the numbers of source and destination antennas are different, and the derived closed-form expressions can be used for determining the sum-power allocation of a relay network. Simulation results confirm that the proposed analytical expressions provide an accurate estimation for the ergodic capacity in various practical environments.

This letter is organized as follows: Section II describes the system model for MIMO AF relay channels. In Section III, we derive new ergodic capacity expressions. Section IV presents the simulation results. Finally, the letter is terminated with conclusions in Section V.

II. SYSTEM MODEL

In this section, we describe a system model for wireless relay networks shown in Fig. 1. The source, the relay, and the

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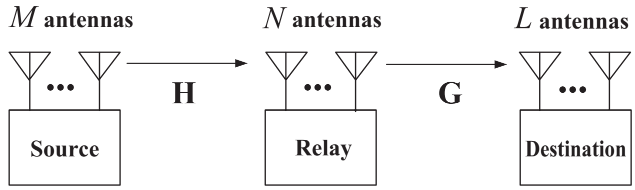

Fig. 1. Schematic diagram of MIMO AF relay networks with M source, N relay, and L destination antennas.

destination node are equipped with M, N and L antennas, respectively, and operated in the time division duplex (TDD) half-duplex mode. In this letter, we assume that a direct link between source and destination can be ignored due to a large path loss. Also, it is assumed that the relay and the destination have full CSI for total network channels, and the source does not know the CSI.

In the first time slot, the source transmits its signals $\mathbf{s}$ to the relay. Then, the N dimensional received signal vector $\mathbf{r}$ at the relay node is given as $\mathbf{r} = \mathbf{Hs} + \mathbf{n}$ where $\mathbf{H}$ is the $N \times M$ channel matrix from the source to the relay, $\mathbf{n}$ denotes the additive complex Gaussian noise vector with zero mean and $\mathcal{E}(\mathbf{nn}^\dagger) = \sigma_n^2 \mathbf{I}_N$, and $\mathbf{s}$ represents the M dimensional transmitted signal vector with $\mathcal{E}(\mathbf{ss}^\dagger) = \frac{P_T}{M}\mathbf{I}_M$. Here, P_T indicates the transmission power at the source, and $\mathbf{I}_N$, $(\cdot)^\dagger$ and $\mathcal{E}$ represent the N dimensional identity matrix, the conjugate transpose and the expectation operator, respectively.

In the second time slot, the received signal $\mathbf{r}$ is multiplied by the $N \times N$ weighting matrix $\mathbf{F}$. Then, the signal vector $\mathbf{x}$ transmitted from the relay node is written by $\mathbf{x} = \mathbf{Fr} = \mathbf{FHs} + \mathbf{Fn}$ where $\mathbf{F}$ should satisfy the relay power constraint $\mathcal{E}\{\mathrm{Tr}(\mathbf{xx}^\dagger)\} \leq P_R$ which can be expressed as

$$\mathrm{Tr}\left\{\mathbf{F}\left(\frac{P_T}{M}\mathbf{HH}^\dagger + \sigma_n^2 \mathbf{I}_N\right)\mathbf{F}^\dagger\right\} \leq P_R. \tag{1}$$

Here $\mathrm{Tr}(\mathbf{A})$ denotes the trace of a matrix $\mathbf{A}$. The processed signal $\mathbf{x}$ is transmitted from the relay node to the destination node. Then the received signal at the destination node $\mathbf{y}$ can be denoted by

$$\mathbf{y} = \mathbf{GFHs} + \mathbf{GFn} + \mathbf{z} \tag{2}$$

where $\mathbf{G}$ is the $L \times N$ channel matrix between the relay and the destination, and $\mathbf{z}$ is the complex white Gaussian noise vector with zero mean and the covariance matrix $\sigma_z^2 \mathbf{I}_L$. Also, elements of the channel matrices $\mathbf{H}$ and $\mathbf{G}$ have independent and identically distributed (i.i.d.) complex Gaussian distribution with zero mean and unit variance.

It was shown in [6] and [7] that the SVD-based linear processing achieves the capacity of MIMO relay channels as in point-to-point MIMO systems. The capacity achieving relay matrix $\mathbf{F}$ can be written as [6][7]

$$\mathbf{F} = \mathbf{V}_G \mathbf{\Sigma}_F \mathbf{U}_H^\dagger \tag{3}$$

where $\mathbf{\Sigma}_F$ is an $N \times N$ diagonal matrix as $\mathbf{\Sigma}_F \triangleq \mathrm{diag}[\sqrt{f_1}, \sqrt{f_2}, \cdots, \sqrt{f_N}]$, and the $N \times N$ unitary matrices $\mathbf{U}_H$ and $\mathbf{V}_G$ are obtained through the SVD of the channels $\mathbf{H} = \mathbf{U}_H \mathbf{\Sigma}_H \mathbf{V}_H^\dagger$ and $\mathbf{G} = \mathbf{U}_G \mathbf{\Sigma}_G \mathbf{V}_G^\dagger$ with the singular values in descending order. Here, f_k of the diagonal matrix $\mathbf{\Sigma}_F$ in (3) is given by [6][7]

$$f_k = \frac{\sigma_z^2 \left[\sqrt{\frac{P_T^2}{M^2\sigma_n^4}\alpha_k^2 + 4\frac{P_T}{M\sigma_n^2}\alpha_k\beta_k\hat{\nu}} - \frac{P_T}{M\sigma_n^2}\alpha_k - 2\right]^+}{2\beta_k\left(\frac{P_T}{M}\alpha_k + \sigma_n^2\right)} \tag{4}$$

where α_k and β_k denote the k-th largest eigenvalues of $\mathbf{HH}^\dagger$ and $\mathbf{G}^\dagger\mathbf{G}$, respectively, $[x]^+$ is defined by $[x]^+ = \max\{0, x\}$, and $\hat{\nu}$ is given as a unique solution of

$$\sum_{k=1}^{\min(M,N,L)} f_k\left(\frac{P_T}{M}\alpha_k + \sigma_n^2\right) = P_R \tag{5}$$

obtained by applying the relay matrix (3) to the relay power constraint (1). It should be noted that we have nonzero f_k only for $k = 1, 2, \cdots, \min(M, N, L)$ in (4).

III. Asymptotic Ergodic Capacity Analysis

In this section, we provide the asymptotic analysis for the ergodic capacity of MIMO AF relaying systems. Assuming that the transmitted signals and the total noise terms in (2) have a zero mean circularly symmetric complex Gaussian distribution, the instantaneous capacity of MIMO AF relay systems can be expressed by

$$C = \frac{1}{2}\log_2\left|\mathbf{I}_L + \frac{P_T}{M}\mathbf{GFHH}^\dagger\mathbf{F}^\dagger\mathbf{G}^\dagger\left(\sigma_n^2\mathbf{GFF}^\dagger\mathbf{G}^\dagger + \sigma_z^2\mathbf{I}_L\right)^{-1}\right|$$

where the pre-log factor $1/2$ is caused by the half-duplex mode, and $|\mathbf{A}|$ indicates the determinant of a matrix $\mathbf{A}$. By applying the capacity optimal relay matrix (3) and taking an expectation over channel realizations, we obtain the ergodic capacity as

$$\mathcal{E}\{C\} = \sum_{k=1}^{\min(M,N,L)} \mathcal{E}\{C_k\} \tag{6}$$

where the ergodic capacity of each stream $\mathcal{E}\{C_k\}$ is defined as

$$\mathcal{E}\{C_k\} \triangleq \frac{1}{2}\mathcal{E}\left\{\log_2\left(1 + \frac{P_T}{M}\frac{f_k\alpha_k\beta_k}{\sigma_n^2 f_k\beta_k + \sigma_z^2}\right)\right\}. \tag{7}$$

Now, the expectation in (7) should be taken over the ordered eigenvalues of Wishart matrices α_k and β_k whose joint distribution functions are given in [12]. However, it is difficult to derive a closed-form solution because of extremely complicated joint distributions. Instead, various asymptotic analyses are possible to obtain a closed-form representation of the ergodic capacity. In this letter, we consider three cases of i) large N, fixed M and L, ii) large M, N and fixed L, and iii) large L, N and fixed M as follows.

A. Large Relay, and Small Source and Destination Antennas

In the following theorem, we derive the ergodic capacity of systems with a large number of relay antennas and fixed numbers of source and destination antennas as a closed-form.

Theorem 1: (Large N, and Fixed M and L) As $N \to \infty$, the ergodic capacity of MIMO AF relay systems obeys $\lim_{N\to\infty}\left[\mathcal{E}\{C\} - \bar{C}_{MNL}\right] = 0$, and equivalently for large

$$\widetilde{C}_{MNL} = \frac{1}{2}\sum_{k=1}^{L}\log_2\left[1+\frac{P_T P_R N\left(b_{MN}\cdot\mu_k + a_{MN}\right)}{\sigma_n^2\sigma_z^2 LM + \sigma_n^2 P_R NM + \sigma_z^2 P_T\sum_{i=1}^{L}\left(b_{MN}\cdot\mu_i + a_{MN}\right)}\right]. \quad (8)$$

but finite N, the ergodic capacity is approximated by $\mathcal{E}\{C\} \approx \bar{C}_{MNL}$ where $\bar{C}_{MNL}$ is given by

$$\bar{C}_{MNL} = \frac{\min(M,L)}{2}\times \log_2\left\{1+\frac{P_T P_R N^2}{\sigma_n^2 P_R MN + \sigma_z^2\min(M,L)(P_T N + \sigma_n^2 M)}\right\}. \quad (9)$$

Proof: As N goes to infinity, we note that $\mathbf{H}^\dagger\mathbf{H}/N$ and $\mathbf{G}\mathbf{G}^\dagger/N$ converge to $\mathbf{I}_M$ and $\mathbf{I}_L$, respectively, since the elements of $\mathbf{H}$ and $\mathbf{G}$ are i.i.d. Gaussian random variables and thus various conditions for strong convergence are readily satisfied [11]. Therefore, for fixed M and L, the asymptotic eigenvalues are given as

$$\alpha_k \longrightarrow N \quad \text{for } k=1,2,\cdots,M, \quad (10)$$
$$\beta_k \longrightarrow N \quad \text{for } k=1,2,\cdots,L. \quad (11)$$

Since all α_k and β_k have the deterministic value N, the relay power coefficients f_k in (4) equal the same value for $k = 1,2,\cdots,\min(M,N,L)$, and thus can be simply computed from the power constraint (5) as

$$f_k = \frac{P_R}{\min(M,L)\left(\frac{P_T}{M}N + \sigma_n^2\right)}. \quad (12)$$

Finally, substituting (10), (11) and (12) to (6) and (7) yields the closed-form ergodic capacity in (9). ■

Since for the case of $M > L$, the ergodic capacity with large N is given as $\mathcal{E}\{C\} \approx \frac{L}{2}\log_2\left\{1+\frac{P_T P_R N^2}{\sigma_z^2 P_T NL + \sigma_n^2(P_R N + \sigma_z^2 L)M}\right\}$ according to Theorem 1, we observe that the ergodic capacity is a decreasing function of M. This implies that it is not helpful to employ larger source antennas than the rank of a relay system $\min(M,N,L)$. This capacity shrink phenomenon is similar to the channel hardening effect found in MIMO multi-user diversity systems [11]. It is worth to note that unlike the channel hardening effect, this capacity shrink is induced when $\mathbf{H}$ has greater rank than $\mathbf{G}$. Although the source node equally distributes the transmit power along all M transmit antennas, the number of active transmit streams are restricted by the rank of $\mathbf{G}$ as shown in (9). Hence, as M becomes larger than L, the source transmit power per stream is reduced and thus the ergodic capacity is decreased.

B. Large Source and Relay, and Small Destination Antennas

Theorem 1 provides a simple closed-form approximation for fixed M and L, but may not match well with empirical results as M or L becomes large. Therefore, in the following theorem, we consider the case of increasing both M and N for fixed L.

Theorem 2: (Large M and N, and Fixed L) As M and $N \to \infty$, the ergodic capacity of MIMO AF relay systems obeys $\lim_{M\to\infty}\lim_{N\to\infty}\left[\mathcal{E}\{C\} - \widetilde{C}_{MNL}\right] = 0$, and equivalently for large but finite M and N, the ergodic capacity is approximated by $\mathcal{E}\{C\} \approx \widetilde{C}_{MNL}$ where $\widetilde{C}_{MNL}$ is given by (8) at the top of this page. Here a_{MN} and b_{MN} are defined as

$$a_{MN} = \left(\sqrt{M}+\sqrt{N}\right)^2, \quad (13)$$
$$b_{MN} = \left(\sqrt{M}+\sqrt{N}\right)\left(\frac{1}{\sqrt{M}}+\frac{1}{\sqrt{N}}\right)^{\frac{1}{3}}, \quad (14)$$

and μ_k is a constant.

TABLE I
μ_k FOR VARIOUS k

μ_1	μ_2	μ_3	μ_4	μ_5
-1.7711	-3.6754	-5.1713	-6.4745	-7.6572
μ_6	μ_7	μ_8	μ_9	μ_{10}
-8.7545	-9.7867	-10.7670	-11.7045	-12.6058

Proof: Since N goes to infinity and L is fixed, $\mathbf{G}\mathbf{G}^\dagger/N$ converges to $\mathbf{I}_L$ as in Theorem 1. Therefore, the eigenvalue of $\mathbf{G}\mathbf{G}^\dagger$ β_k becomes N as in (11). For M and $N\to\infty$, it was shown in [13], [14] and [15] that an appropriately scaled version of the k-th largest eigenvalue of $\mathbf{H}^\dagger\mathbf{H}$ converges to the corresponding Tracy-Widom distribution [9] as

$$p\left(\frac{\alpha_k - a_{MN}}{b_{MN}}\right) \longrightarrow F_2(s,k) \quad (15)$$

where $F_2(s,k)$ is recursively defined by $F_2(s,m+1) = F_2(s,m) + \frac{(-1)^m}{m!}\frac{d^m}{d\lambda^m}D_2(s,\lambda)|_{\lambda=1}$ for $m\geq 0$ with $F_2(s,0) = 0$, $D_2(s,\lambda)$ has the Painlevé representation $D_2(s,\lambda) = \exp\left(-\int_s^\infty (x-s)q^2(x,\lambda)dx\right)$, and $q(x,\lambda)$ is a solution to the Painlevé II equation $q''(x) = xq(x) + 2q(x)^3$ such that $q(x,\lambda)\sim\sqrt{\lambda}\mathrm{Ai}(x)$ as $x\to\infty$ [9].

It is important to note that $F_2(s,k)$ does not depend on M or N. Therefore, denoting the mean and the variance of $F_2(s,k)$ as μ_k and σ_k^2, the mean and the variance of α_k are asymptotically represented from (15) as

$$\mathcal{E}\{\alpha_k\} \longrightarrow b_{MN}\cdot\mu_k + a_{MN}, \quad (16)$$
$$\mathcal{E}\{(\alpha_k - \mathcal{E}\{\alpha_k\})^2\} \longrightarrow b_{MN}^2\cdot\sigma_k^2 \quad (17)$$

where μ_k and σ_k^2 can be calculated as in [16], and it should be noted that μ_k and σ_k^2 for fixed k are finite values. In Table I, μ_k for various k is summarized.

Since in (13) and (14), a_{MN} dominates over b_{MN} as M and N go to infinity, $\frac{\mathcal{E}\{\alpha_1\}}{\mathcal{E}\{\alpha_k\}}$ and $\frac{\mathcal{E}\{(\alpha_k-\mathcal{E}\{\alpha_k\})^2\}}{\mathcal{E}\{\alpha_k\}^2}$ from (16) and (17) converge to one and zero, respectively, due to finite constants μ_k and σ_k^2. This implies that for $k\leq L$, the ratio of the eigenvalues α_1/α_k asymptotically becomes one due to the negligible variance compared to the value of α_k. Since the f_k depends on the ratios of the eigenvalues in (4), f_k becomes a constant value independent of k. Hence, f_k for $k = 1,2,\cdots,L$ satisfying the power constraint (5) is calculated by

$$\widehat{C}_{MNL} = \frac{1}{2}\sum_{k=1}^{M}\log_2\left[1+\frac{P_T P_R N\left(b_{LN}\cdot\mu_k + a_{LN}\right)}{\sigma_z^2 M(P_T N+\sigma_n^2 M)+\sigma_n^2 P_R M\left(b_{LN}\cdot\mu_k+a_{LN}\right)}\right]. \tag{18}$$

$$f_k = \frac{P_R}{\frac{P_T}{M}\sum_{i=1}^{L}\alpha_i + L\sigma_n^2}. \tag{19}$$

Now applying the results of (11) and (19) to (7) yields

$$\begin{aligned}
&\mathcal{E}\{C_k\}\\
&= \frac{1}{2}\mathcal{E}\left\{\log_2\left(1+\frac{P_T P_R N\alpha_k}{\sigma_n^2 P_R NM + \sigma_z^2 P_T\sum_{i=1}^{L}\alpha_i + \sigma_n^2\sigma_z^2 LM}\right)\right\}\\
&= \frac{1}{2}\mathcal{E}\left\{\log_2\mathcal{E}\{\Phi_k\} + \frac{1}{2}\log_2\left(1+\frac{\Phi_k-\mathcal{E}\{\Phi_k\}}{\mathcal{E}\{\Phi_k\}}\right)\right\}\\
&\quad -\frac{1}{2}\mathcal{E}\left\{\log_2\mathcal{E}\{\Omega_k\} + \log_2\left(1+\frac{\Omega_k-\mathcal{E}\{\Omega_k\}}{\mathcal{E}\{\Omega_k\}}\right)\right\}
\end{aligned} \tag{20}$$

where Φ_k and Ω_k are defined as

$$\Phi_k \triangleq \sigma_n^2 P_R NM + \sigma_z^2 P_T\sum_{i=1}^{L}\alpha_i + \sigma_n^2\sigma_z^2 LM + P_T P_R N\alpha_k, \tag{21}$$

$$\Omega_k \triangleq \sigma_n^2 P_R NM + \sigma_z^2 P_T\sum_{i=1}^{L}\alpha_i + \sigma_n^2\sigma_z^2 LM.$$

Using the Maclaurin series in (20), we obtain

$$\begin{aligned}
\mathcal{E}\{C_k\} &= \frac{1}{2}\log_2\mathcal{E}\{\Phi_k\} - \frac{1}{2}\log_2\mathcal{E}\{\Omega_k\}\\
&\quad + \frac{1}{2\ln 2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{\mathcal{E}\{(\Phi_k-\mathcal{E}\{\Phi_k\})^n\}}{\mathcal{E}\{\Phi_k\}^n}\\
&\quad - \frac{1}{2\ln 2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{\mathcal{E}\{(\Omega_k-\mathcal{E}\{\Omega_k\})^n\}}{\mathcal{E}\{\Omega_k\}^n}.
\end{aligned} \tag{22}$$

Here, it is obvious that in the third term of the right hand side, $\frac{\mathcal{E}\{(\Phi_k-\mathcal{E}\{\Phi_k\})\}}{\mathcal{E}\{\Phi_k\}}$ with $n = 1$ equals zero. Also for $n = 2$, $\frac{\mathcal{E}\{(\Phi_k-\mathcal{E}\{\Phi_k\})^2\}}{\mathcal{E}\{\Phi_k\}^2}$ is represented by the variance and the mean of Φ_k. Since Φ_k is a linear equation of α_i for $i = 1, 2, \cdots, L$ in (21), the variance of Φ_k is represented by the variance and covariance terms of α_i. Note that the covariance between α_i and α_j with $i > j$ is smaller than the variance of α_i. Therefore, $\mathcal{E}\{(\Phi_k - \mathcal{E}\{\Phi_k\})^2\}$ and $\mathcal{E}\{\Phi_k\}^2$ are asymptotically expressed as $O\left(\frac{\left(\sqrt{M}+\sqrt{N}\right)^{\frac{8}{3}}}{\left(\sqrt{MN}\right)^{\frac{2}{3}}}\right)$ and $O\left(\left(\sqrt{M}+\sqrt{N}\right)^4\right)$ for large M and N by using the variance and mean of α_i in (16) and (17) with (13) and (14), where $u(x) = O(v(x))$ denotes that $|u(x)/v(x)|$ remains bounded as $x \to \infty$. Hence, we can see $\frac{\mathcal{E}\{(\Phi_k-\mathcal{E}\{\Phi_k\})^2\}}{\mathcal{E}\{\Phi_k\}^2} = O\left(1/\left(\left(\sqrt{M}+\sqrt{N}\right)^{\frac{4}{3}}\left(\sqrt{MN}\right)^{\frac{2}{3}}\right)\right)$ for large M and N. Similarly, it can be shown that $\frac{\mathcal{E}\{(\Phi_k-\mathcal{E}\{\Phi_k\})^n\}}{\mathcal{E}\{\Phi_k\}^n}$ with $n > 2$ and $\frac{\mathcal{E}\{(\Omega_k-\mathcal{E}\{\Omega_k\})^n\}}{\mathcal{E}\{\Omega_k\}^n}$ for all n are negligible. Consequently, (22) follows $\mathcal{E}\{C_k\} \approx \frac{1}{2}\log_2\frac{\mathcal{E}\{\Phi_k\}}{\mathcal{E}\{\Omega_k\}}$, and then we can derive (8) by using (16). ■

It was shown in [17] that the Tracy-Widom law in (15) still converges as $N/M \to \infty$ or $N/M \to 0$. Therefore, it is expected that in Theorem 2 the ergodic capacity can be estimated even with relatively small or large M compared to N, which will be verified in the simulation section. Consequently, the results in Theorem 2 are more accurate compared to Theorem 1.

C. Small Source, and Large Relay and Destination Antennas

We derive a closed-form approximation of the ergodic capacity as the number of relay and destination antennas goes to infinity in the following theorem.

Theorem 3: (Large L and N, and Fixed M) As L and $N \to \infty$, the ergodic capacity of MIMO AF relay systems obeys $\lim_{L\to\infty}\lim_{N\to\infty}\left[\mathcal{E}\{C\} - \widehat{C}_{MNL}\right] = 0$, and equivalently for large but finite L and N, the ergodic capacity is approximated by $\mathcal{E}\{C\} \approx \widehat{C}_{MNL}$ where $\widehat{C}_{MNL}$ is written as (18) at the top of this page. Here a_{LN} and b_{LN} are given by (13) and (14).

Proof: Note that this proof is analogous to the proof of Theorem 2. As N goes to infinity and M is fixed, $\mathbf{H}^\dagger\mathbf{H}$ converges to $N\mathbf{I}_M$ and thus α_k becomes N as in (10). When L and N go to infinity, the eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ β_k converge to the corresponding Tracy-Widom distribution as [13][14][15]

$$p\left(\frac{\beta_k - a_{LN}}{b_{LN}}\right) \longrightarrow F_2(s,k). \tag{23}$$

Then, the asymptotic mean and variance of β_k from (23) are written by $\mathcal{E}\{\beta_k\} \longrightarrow b_{LN}\cdot\mu_k + a_{LN}$ and $\mathcal{E}\{(\beta_k - \mathcal{E}\{\beta_k\})^2\} \longrightarrow b_{LN}^2\cdot\sigma_k^2$.

As shown in the proof of Theorem 2, if N and L go to infinity for a fixed M, the ratio of the eigenvalues β_1/β_k converges to one for $k = 1, 2, \cdots, M$. Then, f_k for $k = 1, 2, \cdots, M$ satisfying the power constraint (5) is given as (12). Finally, by substituting (10) and (12) in (7) and using a similar approach with the proof of Theorem 2 for (23), we can arrive at (18). ■

Note that this theorem is available for smaller or larger L than N similar to Theorem 2. Consequently, Theorems 2 and 3 provide more accurate expressions compared to Theorem 1 for various M and L by exploiting the asymptotic eigenvalue results with Tracy-Widom law. Also, it is worth to note that we can exploit the closed-form approximations to design the system parameters of the relay networks. For example, if a sum-power constraint for the source and relay is considered as in [18], the optimal power allocation can be found by using the derived Theorems. The sum-power constraint is important in battery-limited applications like a sensor network. This example for the power distribution will be explained in detail in the simulation section.

It is straightforward to consider a path-loss model in the derived closed-form expressions since the path-loss parameters

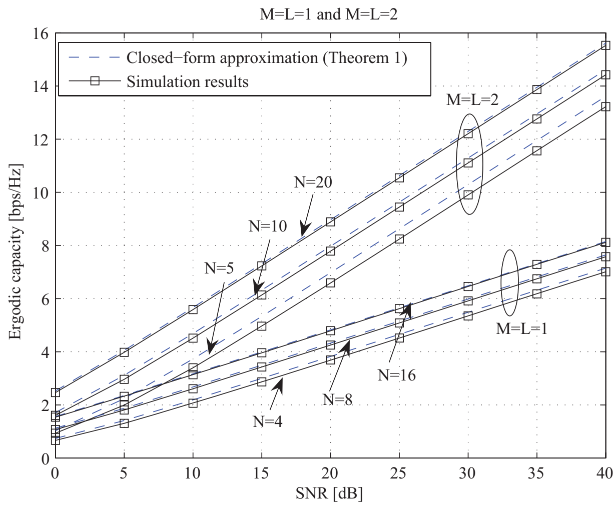

Fig. 2. Comparison of closed-form approximation and simulation results with large N, and fixed M and L.

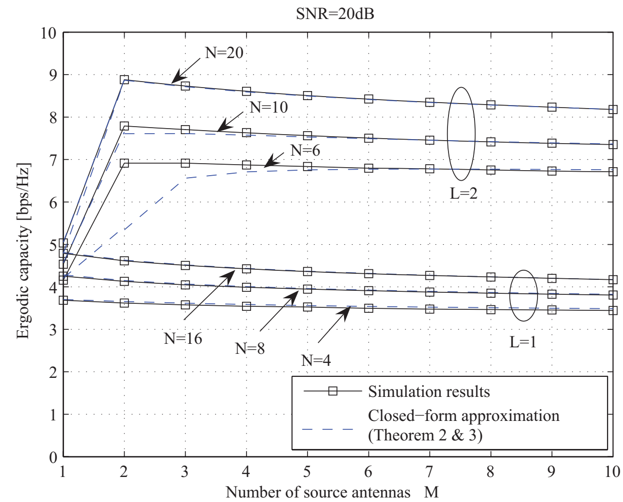

Fig. 3. Comparison of closed-form approximation and Monte Carlo simulation for various number of source antennas M.

of the source-relay and the relay-destination links can be simply represented as the multiplying coefficients of the power constraints P_T and P_R, respectively. Therefore, our theorems can be applicable to select the best relaying path with the geometric information and the shadowing factor in a MIMO relay network. Although we consider the asymptotic case, our analytical results are quite close to the actual simulated results even with the moderate number of antennas thanks to the accuracy of the asymptotic analysis in finite dimension [16], which is demonstrated in the following section.

IV. Simulation Results

In this section, we present simulation results for confirming the analysis of the asymptotic ergodic capacity on MIMO relay channels. In our simulation, we assume that $P_T = P_R = P$ and $\sigma_n^2 = \sigma_z^2 = 1$, and the signal-to-noise ratio (SNR) is defined as P/σ_n^2. All simulation results are plotted by employing the SVD-based capacity achieving scheme in (3) [6][7], and compared with the closed-form approximations in Theorems 1, 2 and 3.

In Fig. 2, we plot the analytical results in Theorem 1 for various SNRs with different number of relay antennas. We can see that as N increases, the approximations match accurately with the simulation results as expected in Theorem 1. It is interesting to see that even for the moderate number of antennas with $N = 4$, our approximation is only a few tenth of a dB away from the empirical results for all SNR range.

Fig. 3 compares the empirical ergodic capacity for various M with the closed-form expression. Here the approximations in Theorem 2 and 3 are used for the cases of $M \geq L$ and $M \leq L$, respectively. Our analysis and simulation results are close in most cases with various M, N and L, although M and N may not be big enough. Also, it is important to see that the capacity is reduced when the number of source antennas M increases for $M > L$. Consequently, these results demonstrate that as expected in Theorem 1, the number of source antennas should be designed not to exceed that of the destination antennas

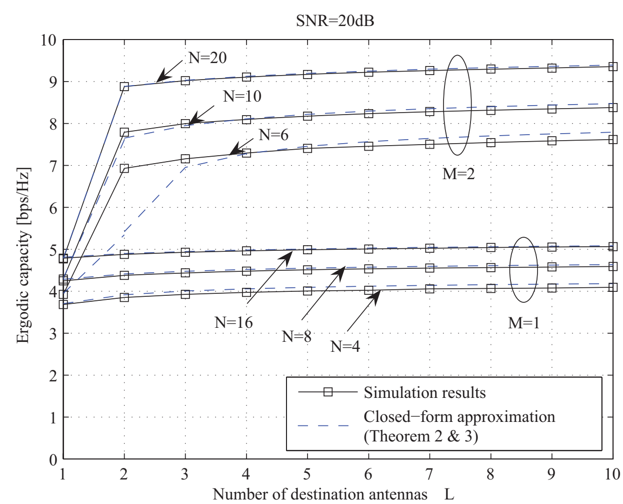

Fig. 4. Comparison of closed-form approximation and Monte Carlo simulation for various number of destination antennas N.

$\mathbf{G}$ (i.e., $M \leq L$) in the case with a large number of relay antennas.

Similarly in Fig. 4, we compare the analyzed results in Theorem 2 and 3 with the corresponding simulation results as the number of destination antennas L increases. For example, our asymptotic analysis with $M = 1$ generates remarkably accurate results for all L. Also, this figure shows that adding the number of destination antennas L is only slightly helpful for the ergodic capacity if $L > \min(M, N, L)$.

Finally, Fig. 5 exhibits the effect of power distribution between the source and the relay nodes, i.e., P_T and P_R. In this figure, it is assumed that the total sum-power constraint is set as $P_{\text{sum}} = P_T + P_R = 6$dB. From both the analytical and empirical results, we can see that the symmetric relay system with $M = L = 2$ is optimized with equal power $P_T = P_R = 3$dB, i.e., the power ratio $P_T/P_R = 1$ as shown in [18]. However, if the destination antennas increase to $L = 10$ and 40, the optimal ergodic capacity is achieved

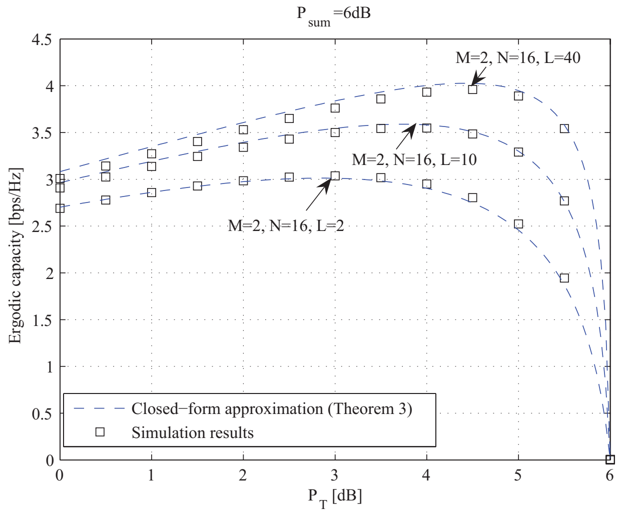

Fig. 5. The effect of power distribution on the sum-power constraint.

at $P_T = 3.8$dB and 4.4dB (i.e., $P_T/P_R = 1.52$ and 2.25), respectively. The optimum power can be obtained by applying simple line search techniques such as the bisection method for the derivative of (18). Consequently, this figure demonstrates that our closed-form analysis is able to accurately determine the optimal power distribution in relay networks.

V. Conclusion

In this letter, we have analyzed the asymptotic ergodic capacity in MIMO AF relay systems with the capacity achieving SVD-based scheme. To obtain the ergodic capacity per stream generated by the SVD, we exploit the asymptotic properties of the eigenvalues of Wishart matrices. Three different asymptotic cases are analyzed. The analyzed ergodic capacity demonstrates a capacity shrink phenomenon which is analogous to the channel hardening effect in MIMO multiuser diversity systems. Hence, it has been found that the number of source antennas do not need to increase more than the number of destination antennas in the case with large relay antennas. Also, we can observe from the analyzed results that the power distribution ratio between source and relay should be optimized according to the number of source, relay, destination antennas. In spite of the asymptotic analysis, simulation results show that our derived analysis generates surprisingly accurate estimation on the ergodic capacity even with a moderate number of antennas and provides helpful insights for designing system parameters in MIMO AF relay networks.

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