Diversity Analysis of Coded Beamforming in MIMO-OFDM Amplify-and-Forward Relaying Systems

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***Abstract*—In this letter, we investigate the diversity performance of coded beamforming schemes in AF multiple antenna relay systems for frequency selective channels. To extract available multipath diversity, we utilize orthogonal frequency division multiplexing combined with bit-interleaved coded modulation. The pairwise error probability is analyzed based on the correlated fading assumption, and the theoretical evaluation of the maximum achievable diversity order is presented. From the analysis, a proper code construction criterion is provided which achieves the full diversity with the minimum code memory. Our analysis also demonstrates that the subcarrier mapping (or pairing) operations at the relay have no impact on the diversity order. Simulations confirm that our analysis is accurate and matches well with the simulation results.**

***Index Terms*—MIMO, relay, diversity, subcarrier pairing.**

I. Introduction

IT has been recognized in recent years that multiple-input and multiple-output (MIMO) wireless systems can potentially provide the improved link performance [1] [2]. More recently, relaying techniques have also garnered a significant interest, since the communication range can be significantly extended. These benefits make MIMO relaying systems a powerful candidate for next generation wireless networks [3]–[6].

One major problem in wireless channels is the signal fluctuation due to fadings. To overcome this fading issue, many diversity techniques have been developed. In flat fading amplify-and-forward (AF) MIMO relaying channels, one of the simplest approaches to achieve full spatial diversity is the optimum beamforming scheme [4] [5] which transmits one data stream through the largest eigenmode in each hop via intelligent use of channel state information. However, for wideband channels, it would still suffer from the frequency selectivity arising from multi-path scatterings. To combat this problem, orthogonal frequency division multiplexing (OFDM) can be used in each hop to transform the frequency selective channel into a set of parallel flat faded subchannels [7]. Recently, many researches on OFDM-based relaying systems have been carried out in [8]–[10] and references therein, but these works are mainly limited to uncoded single antenna scenarios.

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In this letter, we investigate the diversity performance of the optimum beamforming scheme over frequency selective MIMO relaying channels. To exploit additional multipath (or frequency) diversity in wideband channels, we employ the OFDM system combined with the bit-interleaved coded modulation [7]. As subchannels in the OFDM system are normally correlated with one another, the pairwise error probability is analyzed based on the correlated fading assumption and then, the achievable diversity order of the system is established. Based on the analysis, we further present an optimum code construction criterion of the required minimum Hamming distance to achieve the maximum diversity. Our analysis also demonstrates that the subcarrier mapping (or pairing) function at the relay that has been widely studied to enhance the information rate in [8] and [9] has no impact on the diversity order. Finally, simulation results will be presented to confirm the accuracy of the analysis.

Throughout this letter, normal letters represent scalar quantities, boldface letters indicate vectors and boldface uppercase letters designate matrices. The superscript $(\cdot)^T$ and $(\cdot)^*$ stand for the transpose and conjugate transpose, respectively. $\mathcal{E}[\cdot]$ is defined as the expectation operator, and $\mathrm{tr}\,(\mathbf{A})$ and $\|\mathbf{A}\|_F$ represent the trace and the Frobenius norm of a matrix $\mathbf{A}$, respectively. $\sum_{k,d}^{K}$ designates a summation that is taken with index k over the d smallest components among K different terms. $\lceil x \rceil$ equals the smallest integer not less than x. Also, we write $f(x)$ as $o\,(g(x))$ if $f(x)/g(x) \to 0$ as $x \to 0$.

II. System Model

We consider quasi-static frequency selective AF MIMO relaying systems, where the source communicates with the destination via one relay node. The relay node operates in a half duplex mode, and thus the source-to-relay and the relay-to-destination links occupy the same bandwidth. Also no direct path is assumed due to a large path loss between the source and the destination. The transmissions are based on the OFDM modulation, and thus each channel is divided into K flat fading subchannels as shown in Figure 1. A single channel encoder supports all subchannels through a random interleaver [11] which ensures only one bit error in each symbol x_k. We assume perfect time and frequency synchronization among nodes and the inclusion of the cyclic prefix long enough to accommodate the channel delay profile.

After the OFDM modulation and demodulation are performed at both hops, for systems with N_t source, N_r relay and N_d destination antennas, the channel matrices of both hops for the k-th subcarrier are denoted by $\mathbf{H}_k \in \mathbb{C}^{N_r \times N_t}$ and $\mathbf{G}_k \in \mathbb{C}^{N_d \times N_r}$, respectively. Then the (i,j)-th element of

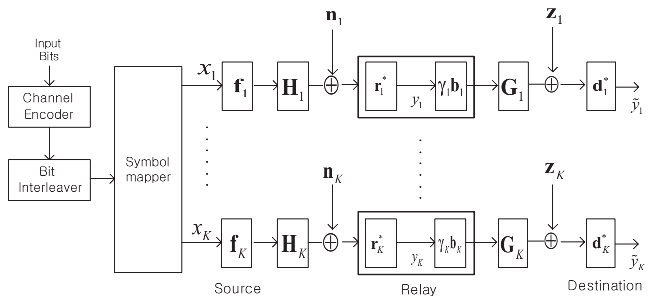

Fig. 1. System description for K parallel OFDM subchannels in coded beamforming AF MIMO relaying systems.

$\mathbf{H}_k$ and $\mathbf{G}_k$ can be expressed as [7]

$$H_k^{i,j} = \sum_{n=1}^{L_h} h^{i,j}(n) e^{-j\frac{2\pi k \tau_n}{KT}} = \mathbf{h}_{i,j}^* \mathbf{w}_k$$
$$\text{and} \quad G_k^{i,j} = \sum_{n=1}^{L_g} g^{i,j}(n) e^{-j\frac{2\pi k \tau_n}{KT}} = \mathbf{g}_{i,j}^* \overline{\mathbf{w}}_k, \quad (1)$$

where T and K are the sampling period and the number of subcarriers, respectively. L_h and L_g denote the number of channel taps of the first hop and the second hop channel, respectively, τ_n indicates the n-th tap delay, and we define $\mathbf{h}_{i,j} = [h^{i,j}(1), h^{i,j}(2), \ldots, h^{i,j}(L_h)]^*$, $\mathbf{g}_{i,j} = [g^{i,j}(1), g^{i,j}(2), \ldots, g^{i,j}(L_g)]^*$, $\mathbf{w}_k = [e^{-j\frac{2\pi k\tau_1}{KT}}, \ldots, e^{-j\frac{2\pi k \tau_{L_h}}{KT}}]^T$ and $\overline{\mathbf{w}}_k = [e^{-j\frac{2\pi k\tau_1}{KT}}, \ldots, e^{-j\frac{2\pi k \tau_{L_g}}{KT}}]^T$. Here we assume that the channel coefficients $h^{i,j}(n)$ and $g^{i,j}(n)$ are independent complex normal Gaussian for all i, j and n. Note that both $H_k^{i,j}$ and $G_k^{i,j}$ are also Gaussian, but correlated in frequency.

In this work, we consider the optimum single stream beamforming scheme [4] [5] in each subcarrier. Also, we assume the equal power and bit allocation over all subcarriers to simplify the analysis. Then, the received signal at the destination for the k-th subchannel is written as

$$\begin{aligned}\tilde{y}_k &= \mathbf{d}_k^*(\gamma_k \mathbf{G}_k \mathbf{b}_k y_k + \mathbf{z}_k) \\ &= \gamma_k \mathbf{d}_k^* \mathbf{G}_k \mathbf{b}_k \mathbf{r}_k^* \mathbf{H}_k \mathbf{f}_k x_k + \gamma_k \mathbf{d}_k^* \mathbf{G}_k \mathbf{b}_k \mathbf{r}_k^* \mathbf{n}_k + \mathbf{d}_k^* \mathbf{z}_k, \quad (2)\end{aligned}$$

where y_k is defined as $y_k \triangleq \mathbf{r}_k^*(\mathbf{H}_k \mathbf{f}_k x_k + \mathbf{n}_k)$, $\mathbf{n}_k$ and $\mathbf{z}_k$ designate the complex normal Gaussian noise vector at the relay and the destination, respectively, $\mathbf{f}_k$ and $\mathbf{b}_k$ indicate the source and the relay transmit beamformer, respectively and $\mathbf{r}_k$ and $\mathbf{d}_k$ denote the relay and the destination receive beamformer, respectively. Each beamformer can be found in [4] or more generally in [5]. In this letter, we follow the flow in [5] to simplify the presentation. Moreover, without loss of generality, we assume $\|\mathbf{f}_k\|^2 = \|\mathbf{b}_k\|^2 = 1$, and thus the per carrier power constraints at the source and the relay are given by $\mathcal{E}[|x_k|^2] = P_T$ and $\mathcal{E}[|\gamma_k y_k|^2] = P_R$ for all k. Then the relay power normalizing coefficient γ_k is determined by $\gamma_k = \sqrt{P_R/\mathcal{E}[y_k y_k^*]}$. Also denoting P_0 as the total transmit power in each subchannel, we assume that $P_T = P_R = P_0/2$. Note that in this case, γ_k is approximated to 1 at high signal-to-noise ratio (SNR).

III. Mathematical Preliminaries

In this section, we present several essential results that will be exploited in our analysis. The following lemma provides a useful rearrangement inequality.

Lemma 1: Let $\mathcal{A} = \{a_1, a_2, \ldots, a_K\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_K\}$ be sets whose elements are positive real values and randomly arranged. We also denote $a_{[i]}$ and $b_{[i]}$ as the i-th smallest component of $\mathcal{A}$ and $\mathcal{B}$, respectively, i.e., $a_{[1]} \leq \ldots \leq a_{[K]}$ and $b_{[1]} \leq \ldots \leq b_{[K]}$. Then we establish the following inequality as

$$\sum_{k=1}^{d} \min(a_{[d-k+1]}, b_{[k]}) \leq \sum_{k,d}^{K} \min(a_k, b_k) \leq \sum_{k=1}^{d} \min(a_{[k]}, b_{[k]}).$$

Proof: See Appendix A. ■

In the subsequent lemma, we derive the approximated probability density function (PDF) of the minimum function of K positive random variables.

Lemma 2: For K positive random variables $\{X_i\}_{i=1,\ldots,K}$[1], we define W as $W \triangleq \min(X_1, X_2, \ldots, X_K)$. Then near the origin ($w \to 0^+$), the PDF $f_W(w)$ of W is given by

$$f_W(w) = f_{X_1}(w) + f_{X_2}(w) + \cdots + f_{X_K}(w),$$

where $f_{X_k}(\cdot)$ for $k = 1, \ldots, K$ indicates the PDF of X_k.

Proof: See Appendix B. ■

IV. Pairwise Error Probability Evaluation

In this section, we provide pairwise error probability (PEP) derivations. As beamformers in each subchannel are designed for minimizing the cost function[2] $\mathcal{E}[|\gamma_k^{-1}\tilde{y}_k - x_k|^2]$, the maximum likelihood decoder at the destination makes a decision according to the rule as $\hat{\mathbf{c}} = \arg\min_{\mathbf{c}\in\mathbb{C}} \sum_{k=1}^{K} |\gamma_k^{-1}\tilde{y}_k - x_k|^2/\sigma_k^2$ where σ_k^2 represents the variance of the effective noise $\gamma_k \mathbf{d}_k^* \mathbf{G}_k \mathbf{b}_k \mathbf{r}_k^* \mathbf{n}_k + \mathbf{d}_k^* \mathbf{z}_k$ in (2), which can be approximated at high SNR as $\sigma_k^2 = \lambda_{h,k}^{-1} + \lambda_{g,k}^{-1}$ [5]. Here $\lambda_{h,k}$ and $\lambda_{g,k}$ mean the largest eigenvalue of $\mathbf{H}_k \mathbf{H}_k^*$ and $\mathbf{G}_k \mathbf{G}_k^*$, respectively.

Now, given $\mathbf{H}_k$ and $\mathbf{G}_k$ for all k, we can consider the conditional PEP that the erroneous coded bit sequence $\bar{\mathbf{c}}$ is chosen over the correct coded bit sequence $\mathbf{c}$ as

$$\begin{aligned} &P(\mathbf{c} \to \bar{\mathbf{c}} | \mathbf{H}_k, \mathbf{G}_k, \forall k) \\ &\quad = P\Big(\sum_{k=1}^{K} \sigma_k^{-2} \left(|\gamma_k^{-1}\tilde{y}_k - x_k|^2 - |\gamma_k^{-1}\tilde{y}_k - \bar{x}_k|^2 \right) > 0 \Big) \\ &\quad = Q\left(\left(\frac{P_T}{2} \sum_{k=1}^{K} \frac{\lambda_{h,k}\lambda_{g,k}}{\lambda_{h,k} + \lambda_{g,k}} d_k^2 \right)^{1/2} \right), \quad (3) \end{aligned}$$

where d_k^2 denotes the normalized Euclidean distance $|x_k - \bar{x}_k|^2/P_T$. Then, denoting d_{free} as the minimum Hamming distance between two codewords, we consider the worst case where d_k is zero for all k except for the d_{free} terms inside the summation in (3). Accordingly, employing a Chernoff

[1] Here, the random variables $X_1, \ldots, X_K$ do not need to be independent.

[2] For single stream cases, minimizing the mean squared error [5] is exactly the same problem with maximizing the received SNR [4]

bound and a harmonic mean inequality $\min(\lambda_{h,k}, \lambda_{g,k}) \geq \frac{\lambda_{h,k}\lambda_{g,k}}{\lambda_{h,k}+\lambda_{g,k}} \geq \min(\lambda_{h,k}, \lambda_{g,k})/2$, the PEP is upperbounded by

$$P(\mathbf{c} \to \bar{\mathbf{c}}|\mathbf{H}_k, \mathbf{G}_k, \forall k) \leq \exp\Big(- \frac{P_0 d_{\min}^2}{16} \sum_{k, d_{\text{free}}}^{K} \min\left(\lambda_{h,k}, \lambda_{g,k}\right)\Big), \quad (4)$$

where $d_{\min}^2$ is the minimum of all nonzero d_k^2. It is well known that Chernoff bounds are tight in the exponential sense for high SNR. In addition, the upper and lower bounds of the harmonic mean are different only by a multiplicative constant factor 2. Therefore the PEP bound in (4) is tight with respect to the diversity order.

Now, let us define the channel gain μ as $\mu \triangleq \sum_{k,d_{\text{free}}}^{K} \min\left(\lambda_{h,k}, \lambda_{g,k}\right)$. Then, using the rearrangement inequality in Lemma 1, μ is bounded as

$$\sum_{k=1}^{d_{\text{free}}} \min(\lambda_{h,[d_{\text{free}}-k+1]}, \lambda_{g,[k]}) \leq \mu \leq \sum_{k=1}^{d_{\text{free}}} \min(\lambda_{h,[k]}, \lambda_{g,[k]}), \quad (5)$$

where $\lambda_{h,[i]}$ and $\lambda_{g,[i]}$ designate the i-th smallest element of the sets $\{\lambda_{h,1}, \ldots, \lambda_{h,K}\}$ and $\{\lambda_{g,1}, \ldots, \lambda_{g,K}\}$, respectively. To compute the average PEP, we need to find the PDF $f_\mu(\mu)$ of μ. However, a direct evaluation of the PDF is quite difficult, due to randomly arranged and correlated eigenvalues in each hop. Therefore, in this letter, we solve the problem through the tight upper and lower bounds of μ in (5).

The upper bound of μ in (5) is actually encountered when the *ordered subcarrier pairing* (OSP) function [8] [9] is employed at the relay. On the contrary, the lower bound holds when the *reversely ordered subcarrier pairing* (R-OSP) scheme [8] [9] is adopted at the relay. By showing that both the OSP and the R-OSP achieve the same diversity order, we arrive at the following theorem which is the main contribution of this letter.

Theorem 1: The diversity order of the multi-carrier coded beamforming over frequency selective AF MIMO relaying channels is given by

$$D = N_r \min\big(N_t \min(L_h, d_{\text{free}}), N_d \min(L_g, d_{\text{free}})\big),$$

regardless of the subcarrier pairing algorithms.

Proof: See Appendix C. ■

We now see that for given channel parameters, the maximum achievable diversity is given by $D_{\max} = N_r \min(L_h N_t, L_g N_d)$ with sufficiently large d_{free}. It is also worthwhile to note that uncoded (i.e., $d_{\text{free}} = 1$) or flat fading (i.e., $L_h = L_g = 1$) systems can be characterized as special cases in our analysis as $D = N_r \min(N_t, N_d)$.

V. Discussion

Our analytical result in Theorem 1 provides valuable insights on the relation between the code construction and the diversity order of AF relaying systems for frequency selective channels. Let us consider two cases: $d_{\text{free}} \geq \max(L_h, L_g)$ and $d_{\text{free}} \leq \min(L_h, L_g)$. The first case occurs when both channels do not contain enough diversity in frequency. In this case, the system performance is mostly limited by the minimum delay spread, especially for single antenna systems (i.e. $D = \min(L_h, L_g)$). The second case represents the channel with large delay spreads. As an extreme case, the channel frequency response $H_k^{i,j}$ and $G_k^{i,j}$ at different subcarriers are fully uncorrelated when $L_h = L_g = K$. In this case, d_{free} significantly affects the performance because the diversity order is determined by $D = d_{\text{free}} N_r \min(N_t, N_d)$.

TABLE I
THE SMALLEST d_{free} TO ACHIEVE $D_{\max}$

	$N_t > N_d$	$N_t < N_d$	$N_t = N_d$
$N_t L_h > N_d L_g$	L_g	$\lceil \frac{L_g N_d}{N_t} \rceil$	L_g
$N_t L_h < N_d L_g$	$\lceil \frac{L_h N_t}{N_d} \rceil$	L_h	L_h
$N_t L_h = N_d L_g$	L_g	L_h	$L_h = L_g$

One more interesting case is when the source-to-relay link does not experience sufficient frequency diversity while the relay-to-destination link has large delay spreads, i.e., $L_h < d_{\text{free}} < L_g$, or vice versa. This case accounts for a circumstance where a relay node helps the communication between the outdoor base-station and a user in a confined and shadowed area such as a subway station. Then the diversity order is given by $N_r \min(L_h N_t, d_{\text{free}} N_d)$ and we can achieve the maximum diversity $L_h N_t N_r$ with sufficiently large d_{free}. In this case, from the diversity order point of view, the use of multiple antennas at the destination might not be useful, since the diversity is independent of N_d. However, this is not an indication that a small N_d is always the best choice, because a smaller N_d leads to the larger d_{free} requirement. This tradeoff is well described in Table I which lists the smallest d_{free} to achieve the maximum diversity $D_{\max}$ for various channel configurations. For instance, if we have $L_h \ll L_g$ and $N_t > N_d$, the optimum free distance is given by $d_{\text{free}} = \lceil \frac{L_h N_t}{N_d} \rceil$, which is inversely proportional to N_d. Note that these results are also applicable to many practical relaying strategies such as relay selection methods. For example, among several relays with different channel profiles, we can choose the best one in terms of the diversity order for given antenna and code structures.

VI. Numerical Results

In this section, we present computer simulations to support the analytical result derived in this letter. For OFDM modulation, $K = 64$ subcarriers are used with QPSK and the cyclic prefix of length $\max(L_h, L_g) - 1$ is added to avoid intercarrier interference. A random interleaver is used for simulations, which transmits consecutive coded bits over different subchannels. Instead of high rate punctured codes, we adjust d_{free} by changing the code memory length with a fixed rate $1/2$ for maintaining the same spectral efficiency. The channel in each hop has exponentially decaying mutipath Rayleigh fadings. Also we employ the notation $N_t \times N_r \times N_d$ to denote a system with N_t source, N_r relay and N_d destination antennas. Specifically, we investigate the numerical results according to the following two scenarios of $L_h = L_g = L$ and $L_h < L_g$. The latter case also accounts for $L_h > L_g$ by the symmetric property of the system. For all figures, we include the uncoded case with BPSK that serves as a benchmark to validate our analysis.

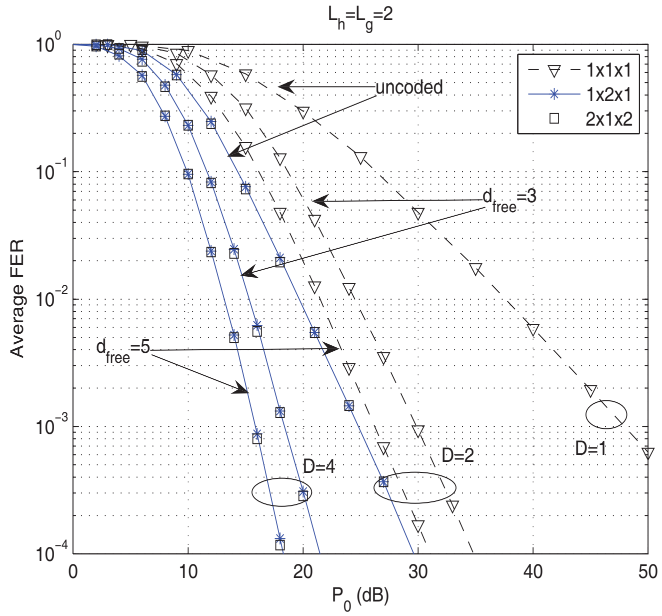

Fig. 2. Average FER performance as a function of the total transmit power P_0 with various system configurations.

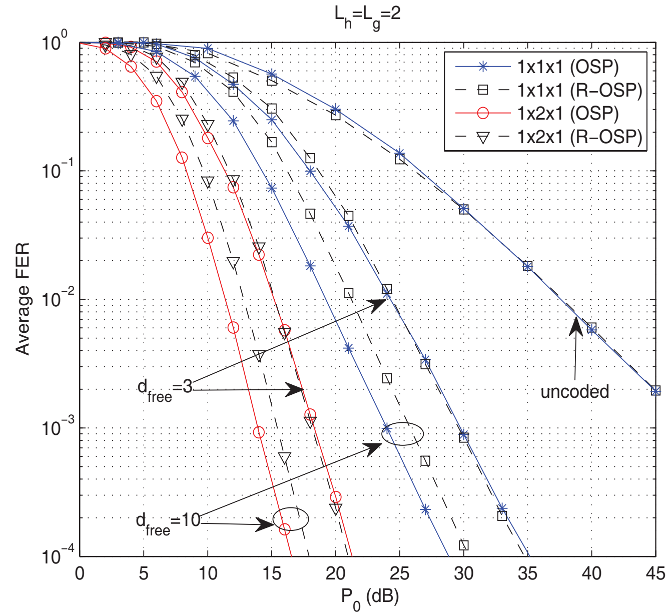

Fig. 4. Average FER comparison with R-OSP and OSP as a function of the total transmit power P_0.

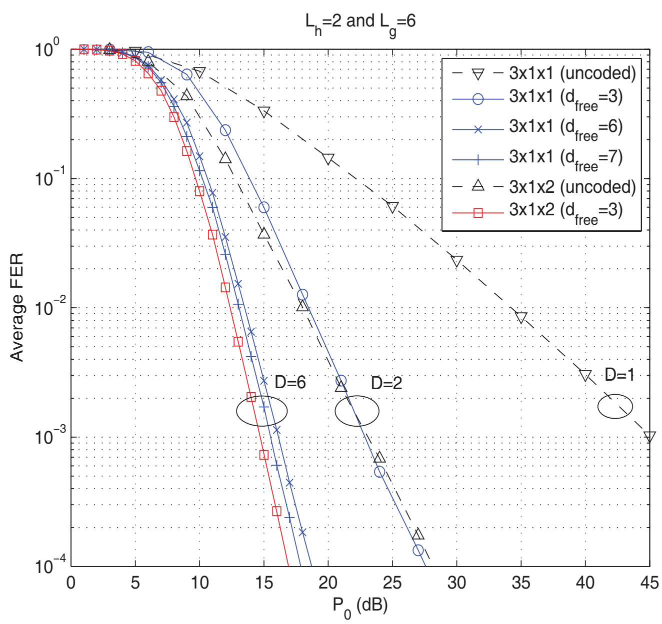

Fig. 3. Average FER performance as a function of the total transmit power P_0 with various system configurations.

First, in Figure 2, we present Monte Carlo simulations for various antenna configurations for $L_h = L_g = 2$. In this case, the diversity is written by $D = N_r \min(2, d_{\text{free}}) \min(N_t, N_d)$ and we see that the derived diversity expression precisely reflects the slope of the frame error rate (FER) curves. In addition, it can be seen from this figure that the diversity order is saturated at $d_{\text{free}} = L$, i.e., when $d_{\text{free}} > L$, a further increment of d_{free} does not introduce any diversity advantage. Also, we can check from this plot that $1 \times 2 \times 1$ and $2 \times 1 \times 2$ systems exhibit the identical performance. This observation leads to the fact that when $L_h = L_g$, increasing N_r would be a more efficient way to achieve the spatial diversity compared to increasing both N_t and N_d.

In Figure 3, numerical simulations for the case of $L_h < L_g$ are illustrated with $L_h = 2$ and $L_g = 6$. As expected, the maximum diversity of both cases with $3 \times 1 \times 1$ and $3 \times 1 \times 2$ is given by $L_h N_t N_r = 6$, which is independent of N_d. In contrast to the case of $L_h = L_g$, we see from the $3 \times 1 \times 1$ system that the use of multiple antennas only at the source can achieve the spatial diversity. It is also clear from this plot that the case of $N_d = 1$ requires the minimum Hamming distance at least $d_{\text{free}} = 6$ to achieve the maximum diversity, while only $d_{\text{free}} = 3$ is sufficient for $N_d = 2$. This observation confirms our analysis that the increased N_d results in the lower d_{free} when $L_h < L_g$.

Figure 4 illustrates the FER performance of two specific pairing schemes OSP and R-OSP which correspond to the channel gain upper and lower bounds in (5), respectively. From this figure, we see that both schemes achieve the same diversity order, which confirms our observation made in the analysis that a pairing algorithm does not affect the diversity order. However, it is interesting to see that as the channel coding gets stronger (i.e. d_{free} becomes larger), the OSP obtains a higher coding gain advantage than the R-OSP. The reason for this is that as d_{free} increases, the gap between two bounds in (5) grows. Note that the gap is tight at $d_{\text{free}} = 1$. This phenomenon is also parallel to the result in [8] and [9] that the OSP outperforms the R-OSP in terms of the mutual information.

VII. Conclusion

In this letter, we have investigated the diversity order of the coded beamforming scheme combined with multi-carrier modulations in wideband MIMO relay channels. Based on the PEP analysis, we have derived the achievable diversity, which is independent of the subcarrier pairing algorithm. Also, we have presented a proper code construction criterion for the required d_{free} that achieves the full diversity order. Simulations results have been reported to confirm the accuracy of the analysis.

APPENDIX

A. Proof of Lemma 1

First, we prove the upper bound using the mathematical induction. Let $\mu_k = \min(a_k, b_k)$. Then we have $\sum_{k,d}^{K} \min(a_k, b_k) = \sum_{k=1}^{d} \mu_{[k]}$. Now, it can be easily seen that $\mu_{[1]}$ equals $\min(a_{[1]}, b_{[1]})$ for all pairing cases, and thus $\mu_{[2]}$ is determined by $\mu_{[2]} = \min\left(\max(a_{[1]}, b_{[1]}), \min(a_{[2]}, b_{[2]})\right)$. This result implies that $\mu_{[2]} \leq \min(a_{[2]}, b_{[2]})$. Now, suppose that $\mu_{[i]} = \min(a_{[i]}, b_{[i]})$ for an arbitrary i. Then as a corollary, we can show that $\mu_{[i+1]}$ cannot exceed $\min(a_{[i+1]}, b_{[i+1]})$. Hence, we finally obtain $\sum_{k=1}^{d} \mu_{[k]} \leq \sum_{k=1}^{d} \min(a_{[k]}, b_{[k]})$. Next, the lower bound can be easily proved, since $\sum_{k=1}^{d} \min(a_{[d-k+1]}, b_{[k]})$ equals the sum of the d smallest components of the set $\mathcal{A} \cup \mathcal{B}$, which is definitely the minimum of all possible pairing sum cases, and the proof is completed.

B. Proof of Lemma 2

Let us define Q_j as $Q_j = \min(X_j, X_{j+1}, \ldots, X_K)$ for $j = 2, \ldots, K-1$. Then W can be rewritten as $W = \min(X_1, Q_2)$. Here we define a dummy variable Z as $Z \triangleq \max(X_1, Q_2)$. Then, the joint PDF of W and Z for $w \leq z$ can be obtained as [12] $f_{W,Z}(w, z) = f_{X_1,Q_2}(w, z) + f_{X_1,Q_2}(z, w)$. Note that when $w > z$, $f_{W,Z}(w, z)$ equals 0, because there is no such a case. Taking the integration on the joint PDF over Z, the marginal PDF of W is written by $f_W(w) = \int_w^\infty f_{X_1,Q_2}(w, z)dz + \int_w^\infty f_{X_1,Q_2}(z, w)dz$. Especially near the origin ($w \to 0^+$), it can be approximated as $f_W(w) = f_{X_1}(w) + f_{Q_2}(w)$. In a similar way, we can express $f_{Q_j}(w)$ as $f_{Q_j}(w) = f_{X_j}(w) + f_{Q_{j+1}}(w)$ near the origin. Therefore, finally we have $f_W(w) = \sum_{k=1}^{K} f_{X_k}(w)$, and we complete the proof.

C. Proof of Theorem 1

We prove Theorem 1 by showing both the upper and lower bounds in (5) are tight in terms of the diversity order. Let us first introduce the moment generating function of a sum of K independent Gamma random variables as

$$\mathcal{L}_U(s) \triangleq \int_0^\infty e^{-su} f_U(u) du = \prod_{i=1}^{K} \frac{1}{(1 + s/\beta_i)^{\alpha_i}}, \tag{6}$$

where $U = Z_1 + Z_2 + \cdots + Z_K$ with $Z_i \backsim \mathcal{G}(\alpha_i, \beta_i) \triangleq (\beta_i^{\alpha_i}/\Gamma(\alpha_i))\, z_i^{\alpha_i - 1} e^{-\beta_i z_i}$ for $i = 1, \ldots, K$. This result will be exploited later in deriving the average PEP.

Now, in the following, we evaluate the diversity order achieved by the channel gain upper bound in (5). Assuming that μ equals its upper bound and applying the following inequality[3]

$$\sum_{k=1}^{d_{\text{free}}} \min(\lambda_{h,[k]}, \lambda_{g,[k]}) \geq d_{\text{free}}^{-1} \min\Big(\sum_{k=1}^{d_{\text{free}}} \lambda_{h,[k]}, \sum_{k=1}^{d_{\text{free}}} \lambda_{g,[k]}\Big),$$

the PEP in (4) can be rewritten as

$$\begin{aligned} &P(\mathbf{c} \to \bar{\mathbf{c}} | \mathbf{H}_k, \mathbf{G}_k, \forall k) \\ &\quad \leq \exp\left(-\frac{P_0 d_{\min}^2}{16 d_{\text{free}}} \min\left(\sum_{k=1}^{d_{\text{free}}} \lambda_{h,[k]}, \sum_{k=1}^{d_{\text{free}}} \lambda_{g,[k]}\right)\right). \end{aligned} \tag{7}$$

Denoting N as $N = \max(N_t, N_r, N_d)$, we get $\lambda_{h,k} \geq \frac{1}{N}\text{tr}(\mathbf{H}_k \mathbf{H}_k^*) = \frac{1}{N}\|\mathbf{H}_k\|_F^2$. Thus, we can establish another tight bound $\sum_{k=1}^{d_{\text{free}}} \lambda_{h,[k]} \geq U_{h,d_{\text{free}}}$ where $U_{h,d_{\text{free}}} \triangleq \frac{1}{N}\sum_{k=1}^{d_{\text{free}}} \|\mathbf{H}_{[k]}\|_F^2$. Then, following the definition in (1), $U_{h,d_{\text{free}}}$ can be evaluated as

$$U_{h,d_{\text{free}}} = \frac{1}{N} \sum_{k=1}^{d_{\text{free}}} \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} \left|\mathbf{h}_{i,j}^* \mathbf{w}_{[k]}\right|^2 = \frac{1}{N} \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} \mathbf{h}_{i,j}^* \mathbf{E} \mathbf{h}_{i,j},$$

where $\mathbf{E} \triangleq \sum_{k=1}^{d_{\text{free}}} \mathbf{w}_{[k]} \mathbf{w}_{[k]}^*$ is a Hermitian matrix whose rank is $\min(L_h, d_{\text{free}})$ and by the eigen decomposition, $\mathbf{E}$ is given by $\mathbf{E} = \mathbf{P}^* \mathbf{\Phi} \mathbf{P}$ where $\mathbf{\Phi}$ is a diagonal matrix with non-zero entries $\phi_{h,1}, \ldots, \phi_{h,\min(L_h, d_{\text{free}})}$.

Then it follows

$$U_{h,d_{\text{free}}} = \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} \sum_{l=1}^{\min(L_h, d_{\text{free}})} \frac{\phi_{h,l}}{N} |\hat{h}^{i,j}(l)|^2,$$

where $\hat{h}^{i,j}(l)$ indicates the l-th element of $\mathbf{P}\mathbf{h}_{i,j}$. Since $\mathbf{P}$ is unitary, $\mathbf{P}\mathbf{h}_{i,j}$ has the same statistical property with $\mathbf{h}_{i,j}$. Thus $\frac{\phi_{h,l}}{N}|\hat{h}^{i,j}(l)|^2 \backsim \mathcal{G}(1, N/\phi_{h,l})$. Similarly, we have $\sum_{k=1}^{d_{\text{free}}} \lambda_{g,[k]} \geq U_{g,d_{\text{free}}}$ where $U_{g,d_{\text{free}}} = \sum_{l=1}^{\min(L_g, d_{\text{free}})} \sum_{i=1}^{N_d} \sum_{j=1}^{N_r} \frac{\phi_{g,l}}{N} |\hat{g}^{i,j}(l)|^2$ with $\frac{\phi_{g,l}}{N}|\hat{g}^{i,j}(l)|^2 \backsim \mathcal{G}(1, N/\phi_{g,l})$ for the second hop channel. This result finally yields

$$\min\Big(\sum_{k=1}^{d_{\text{free}}} \lambda_{h,[k]}, \sum_{k=1}^{d_{\text{free}}} \lambda_{g,[k]}\Big) \geq \hat{\mu} = \min(U_{h,d_{\text{free}}}, U_{g,d_{\text{free}}}). \tag{8}$$

It is known that at high SNR, the average error performance depends only on the limiting behavior of the channel gain near the origin [11]. Thus the approximated PDF $f_{\hat{\mu}}(\hat{\mu})$ of $\hat{\mu}$ is simply given by a sum of PDFs of $U_{h,d_{\text{free}}}$ and $U_{g,d_{\text{free}}}$ from Lemma 2. Moreover, $U_{h,d_{\text{free}}}$ and $U_{g,d_{\text{free}}}$ are equivalent to the sum of $N_t N_r \min(L_h, d_{\text{free}})$ and $N_r N_d \min(L_g, d_{\text{free}})$ independent Gamma random variables, respectively. Accordingly, substituting (8) into the PEP bound in (7) and averaging over $\hat{\mu}$ by applying the integral formula in (6), the average PEP at high SNR is calculated as

$$\begin{aligned} &P(\mathbf{c} \to \bar{\mathbf{c}}) \\ &\leq \int_0^\infty \exp(-\rho \cdot \hat{\mu})\, f_{\hat{\mu}}(\hat{\mu})\, d\hat{\mu} = \mathcal{L}_{U_{h,d_{\text{free}}}}(\rho) + \mathcal{L}_{U_{g,d_{\text{free}}}}(\rho) \\ &= \prod_{l=1}^{\min(L_h, d_{\text{free}})} \left(1 + \frac{\phi_{h,l}\rho}{N}\right)^{-N_t N_r} + \prod_{l=1}^{\min(L_g, d_{\text{free}})} \left(1 + \frac{\phi_{g,l}\rho}{N}\right)^{-N_r N_d} \\ &\simeq \left(\frac{\phi_{\min}}{N}\rho\right)^{-D} + o\left(P_0^{-D}\right). \end{aligned}$$

[3]Note that this inequality is tight in terms of diversity, since we also have $\sum_{k=1}^{d_{\text{free}}} \min(\lambda_{h,[k]}, \lambda_{g,[k]}) \leq \min(\sum_{k=1}^{d_{\text{free}}} \lambda_{h,[k]}, \sum_{k=1}^{d_{\text{free}}} \lambda_{g,[k]})$ by Jensen's inequality [13].

where ρ and $\phi_{\min}$ are defined as $\rho \triangleq \frac{P_0 d_{\min}^2}{16 d_{\text{free}}}$ and the minimum of all non-zero eigenvalues, respectively. Finally, the diversity order D is determined by

$$D = N_r \min\left(N_t \min(L_h, d_{\text{free}}), N_d \min(L_g, d_{\text{free}})\right). \quad (9)$$

Now, we consider the channel gain lower bound in (5). Then, unlike the previous case, the conditional PEP in (4) can be rephrased as

$$\begin{aligned} &P(\mathbf{c} \to \bar{\mathbf{c}} | \mathbf{H}_k, \mathbf{G}_k, \forall k) \\ &\quad \leq \exp\Big(-\frac{P_0 d_{\min}^2}{16} \sum_{k=1}^{d_{\text{free}}} \min(\lambda_{h,[d_{\text{free}}-k+1]}, \lambda_{g,[k]})\Big), \quad (10) \end{aligned}$$

Let Γ be defined by $\Gamma \triangleq \sum_{k=1}^{d_{\text{free}}} \min(\lambda_{h,[d_{\text{free}}-k+1]}, \lambda_{g,[k]})$. Then Γ can be further lower-bounded by the minimum of all its possible outputs. For example, if $d_{\text{free}} = 2$, $\Gamma \geq \min(\lambda_{h,[1]} + \lambda_{h,[2]}, \lambda_{h,[1]} + \lambda_{g,[1]}, \lambda_{g,[1]} + \lambda_{g,[2]})$. Accordingly, using the similar approach in (8), we have

$$\begin{aligned} \Gamma &\geq \min_{i=0,1,\ldots,d_{\text{free}}} \Big(\sum_{k=1}^{d_{\text{free}}-i} \lambda_{h,[k]} + \sum_{k=1}^{i} \lambda_{g,[k]} \Big) \\ &\geq \quad \check{\mu} \quad = \min_{i=0,1,\ldots,d_{\text{free}}} \left(U_{h,d_{\text{free}}-i} + U_{g,i}\right). \quad (11) \end{aligned}$$

Note that $U_{h,0} = U_{g,0} = 0$. Denoting $\check{\mu}_i \triangleq U_{h,d_{\text{free}}-i} + U_{g,i}$, the above bound can be more simplified as $\Gamma \geq \min\left(\check{\mu}_0, \check{\mu}_1, \ldots, \check{\mu}_{d_{\text{free}}}\right)$, and each $\check{\mu}_i$ equals the sum of $N_t N_r \min(L_h, d_{\text{free}} - i) + N_r N_d \min(L_g, i)$ independent Gamma random variables. Therefore, substituting (11) into the bound in (10) and averaging with respect to $\check{\mu}$ by Lemma 2 and the formula in (6), the average PEP is readily obtained as

$$\begin{aligned} &P(\mathbf{c} \to \bar{\mathbf{c}}) \\ &\leq \sum_{i=0}^{d_{\text{free}}} \left(1 + \frac{\phi_{\min}}{N}\rho\right)^{-(N_t N_r \min(L_h, d_{\text{free}}-i) + N_r N_d \min(L_g, i))} \\ &\simeq \left(\frac{\phi_{\min}}{N}\rho\right)^{-D} + o\left(P_0^{-D}\right). \end{aligned}$$

We now see that the diversity order D equals $\min_{i=0,\ldots,d_{\text{free}}} \left(N_t N_r \min(L_h, d_{\text{free}} - i) + N_r N_d \min(L_g, i)\right)$. Also, it is easy to show that the diversity order D given by above bound is the same as (9). From this result, we finally demonstrate that the diversity order achieved by the channel gain μ in (5) is determined by D regardless of the subcarrier pairings, and we conclude the proof.

REFERENCES

[1] H. Lee, B. Lee, and I. Lee, "Iterative detection and decoding with an improved V-BLAST for MIMO-OFDM systems," *IEEE J. Sel. Areas Commun.*, vol. 24, pp. 504–513, Mar. 2006.

[2] H. Lee and I. Lee, "New approach for error compensation in coded V-BLAST OFDM systems," *IEEE Trans. Commun.*, vol. 55, pp. 345–355, Feb. 2007.

[3] B. Wang, J. Zhang, and A. Host-Madsen, "On the capacity of MIMO relay channels," *IEEE Trans. Inf. Theory*, vol. 51, pp. 29–43, Jan. 2005.

[4] B. Khoshnevis, W. Yu, and R. Adve, "Grassmannian beamforming for MIMO amplify-and-forward relaying," *IEEE J. Sel. Areas Commun.*, vol. 26, pp. 1397–1407, Oct. 2008.

[5] C. Song, K.-J. Lee, and I. Lee, "MMSE based transceiver designs in closed-loop non-regenerative MIMO relaying systems," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 2310–2319, July 2010.

[6] K.-J. Lee, H. Sung, E. Park, and I. Lee, "Joint optimmization for one and two-way MIMO AF multiple-relay systems," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 3671–3681, Dec. 2010.

[7] I. Lee, A. Chan, and C.-E. W. Sundberg, "Space-time bit-interleaved coded modulation for OFDM systems," *IEEE Trans. Signal Process.*, vol. 52, pp. 820–825, Mar. 2004.

[8] I. Hammerstrom and A. Wittneben, "Power allocation schemes for amplify-and-forward MIMO-OFDM relay links," *IEEE Trans. Wireless Commun.*, vol. 6, pp. 2798–2802, Aug. 2007.

[9] Y. Li, W. Wang, J. Kong, and M. Peng, "Subcarrier pairing for amplify-and-forward and decode-and-forward OFDM relay links," *IEEE Commun. Lett.*, vol. 13, pp. 209–211, Apr. 2009.

[10] H. A. Suraweera and J. Armstrong, "Performance of OFDM-based dual-hop amplify-and-forward relaying," *IEEE Commun. Lett.*, vol. 11, pp. 726–728, Sep. 2007.

[11] E. Akay, E. Sengul, and E. Ayanoglu, "Bit interleaved coded multiple beamforming," *IEEE Trans. Commun.*, vol. 55, pp. 1802–1811, Sep. 2007.

[12] A. Leon-Garcia, *Probability and Random Processes for Electrical Engineering*, 3rd edition, Addison-Wesley.

[13] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.