Achievable Degrees of Freedom on K-user Y Channels

Kwangwon Lee, *Student Member, IEEE,* Namyoon Lee, and Inkyu Lee, *Senior Member, IEEE*

***Abstract*—In this paper, we consider K-user Y channels where K users simultaneously exchange messages with each other via an intermediate relay. Degrees of freedom (DOF) of Y channels with multiple antennas is not known in general. Investigation of the feasibility conditions of signal space alignment for network coding is an initial step for addressing this open problem. We verify that when user i with M_i antennas sends $K-1$ independent messages to the other users through a relay with N antennas and each message achieves the DOF of d, the total DOF of $dK(K-1)$ is attained if $M_i \geq d(K-1)$, $N \geq \frac{dK(K-1)}{2}$ and $N < \min\{M_i + M_j - d|\forall\ i \neq j\}$. It is accomplished by adopting the signal space alignment for the network coding during both the multiple access phase and the broadcast phase. It is shown that the proposed scheme obtains not only a network coding gain but also an alignment gain in terms of the normalized DOF, as $K \to \infty$. Also for Y channels where all nodes have a single antenna, we show that the DOF of 2 is achieved regardless of the number of users by using the rational dimension framework.**

***Index Terms*—Degrees of freedom, Y channels, interference alignment, network coding.**

I. INTRODUCTION

TWO-WAY communication between two users was initially considered by Shannon [1]. This two-way channel can be extended to a two-way relay channel by adding a relay between the two users. Many researchers have studied two-way relay channels because of their attractive application scenarios in various fields such as cellular and ad-hoc networks [2][3]. In [2], the authors proposed efficient two-way protocols to overcome a spectral efficiency loss in relaying-aided communication systems. The two-way relay channel consists of two phases: the multiple access channel (MAC) phase and the broadcast channel (BC) phase.

First, in the MAC phase, two users simultaneously send their messages to an intermediate relay. Then, during the BC phase, the relay retransmits the information obtained in the MAC phase to two users. By exploiting the knowledge of their own transmitted information, each user is able to cancel self-interference and decode the intended message. In the context of network coding for the two-way relay channel, analog network coding (ANC) [4] and physical-layer network coding (PNC) [5] have been proposed lately. Both schemes allow simultaneous transmission of two users by performing joint detection.

Two-way relay channels have been generalized to different systems such as bi-directional multi-pair message exchange [6]–[9] and multi-directional multi-pair exchange [10]–[13]. For the multi-pair two-way relay channel, a joint demodulate-and-XOR forward relaying scheme was proposed for code-division multiple access systems under interference limited environments in [6]. Furthermore in [7] and [8], the authors characterized the capacity of multi-pair two-way relay channels in the deterministic channel and the Gaussian channel, respectively.

For multi-pair multi-way relay channels where more than two users communicate via a relay in a bi-directional manner, multiple interfering clusters of users were considered in [11] which communicate simultaneously via a relay, where users within the same cluster exchange messages among themselves. Also upper and lower bounds for the capacity were investigated with different relaying schemes. While each user in a cluster has a single common message intended for all the other users in the same cluster in [11], the authors in [12] and [13] take into account more general settings such as multiple independent messages per user. Especially, a new network information flow called a multiple input multiple output (MIMO) Y channel was proposed in [12] where each of three users intends to convey independent unicast messages to the other users via an intermediate relay while receiving two independent messages from the other two users.

As the number of users who wish to communicate in the wireless medium increases, interference becomes a big issue. In order to successfully deal with an interference problem, various signaling methods have been studied. The idea of interference alignment was introduced by Maddah-Ali et al. [14] for MIMO X channels and by Cadambe and Jafar [15] for K-user interference channels. These works have inspired much hope as the savior to the interference problem by aligning multiple interfering signals at each receiver in order to reduce the effective interference.

By appropriately exploiting the concept of the interference alignment and network coding, signal space alignment for network coding (SSA-NC) was proposed to efficiently deal with multiple interference signals in 3-user MIMO Y channels in [12]. The core concept of the SSA-NC is that each user pair who wants to exchange messages cooperatively constructs the transmit beamforming vectors so that two pair signals for the network coding should be aligned within the same spatial

Manuscript received June 24, 2011; revised November 2, 2011; accepted December 19, 2011. The associate editor coordinating the review of this paper and approving it for publication was X. Xiang-Gen.

K. Lee and I. Lee are with the School of Electrical Engineering, Korea University, Seoul, Korea (e-mail: {kwangwonlee, inkyu}@korea.ac.kr).

N. Lee is with the University of Texas, Austin, USA (e-mail: namyoon.lee@gmail.com).

The material in this paper was presented in part at the IEEE ICC, Kyoto, Japan, June 2011. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2010-0017909).

Digital Object Identifier 10.1109/TWC.2012.012712.111194

signal dimension. Using this scheme, it was shown that the optimal degrees of freedom (DOF) of $3M$ is achieved when each user has M antennas and the relay has $N \geq \lceil \frac{3M}{2} \rceil$ antennas [13]. The DOF is conventionally represented by the pre-log term in the capacity, and is a crucial metric for characterizing the capacity behavior in high signal-to-noise-ratio (SNR) regime for multiple antenna systems [16].

In this paper, we extend the result of [13] into the case of a general number of users as a K-user Y channel and investigate the achievable DOF of this channel. For the multiple antenna case, we first study the feasibility condition of the SSA-NC for the K-user MIMO Y channel in order to obtain the DOF of $dK(K-1)$ when each user transfers $K-1$ independent messages to the other users and each message achieves the DOF of d. To be specific, we answer the question of how many antennas are required in minimum to obtain the DOF of $dK(K-1)$ for K-user MIMO Y channels. To this end, we will modify the SSA-NC scheme in [13] so that signal space alignment is performed not only at the MAC phase but also at the BC phase. In order to show the efficiency of the proposed scheme in terms of the number of antennas, we consider the DOF which is normalized by the average number of node antennas. Subsequently, for the single antenna case, we will show that the DOF of 2 is achievable for K-user single-input single-output (SISO) Y channels by using the rational dimension framework introduced in [17] and [18].

This paper is organized as follows: Section II describes the system model of K-user Y channels. In Section III, we consider multiple antenna systems for both symmetric and asymmetric cases. The feasibility condition to achieve the DOF of $dK(K-1)$ for the K-user MIMO Y channel is investigated using an AF relaying strategy combined with SSA-NC. In Section IV, we study the single antenna case and show that the DOF of 2 is achieved by using the rational dimension framework. Section V verifies the derived results through numerical simulations. Finally, the paper is terminated with conclusions in Section VI.

We use bold upper and lower case letters for matrices and column vectors, respectively. $(\cdot)^T$ and $(\cdot)^H$ represent transpose and conjugate transpose, respectively. Also, $\mathbb{E}(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the expectation and the trace operator of a matrix. We indicate the null space of a matrix $\mathbf{A}$ as $\mathrm{null}(\mathbf{A})$ and the Frobenius norm of $\mathbf{A}$ is defined as $\|\mathbf{A}\|^2 = \mathrm{tr}(\mathbf{A}\mathbf{A}^H)$

II. System Model

The system model for K-user Y channels is shown in Fig. 1. In this channel, user i with M_i antennas wants to send $K-1$ independent messages $W^{[j,i]}$ to user $j \in \{1, 2, \ldots, K\}/\{i\}$ and intends to decode all the other users' messages $\hat{W}^{[i,j]}$ using an intermediate relay with N antennas. All users wish to achieve the DOF of d for each message, i.e., they want to transfer each message with d interference-free streams $\left(d \leq \lfloor \frac{M_i}{K-1} \rfloor\right)$. For simplicity, direct links between users are neglected.

During the MAC phase, user i transmits the signal vector $\mathbf{x}^{[i]}$ precoded as $\mathbf{x}^{[i]} = \sum_{j=1, j\neq i}^{K} \mathbf{V}^{[j,i]}\mathbf{s}^{[j,i]}$, where $\mathbf{V}^{[j,i]}$ represents the $M_i \times d$ beamforming matrix $\mathbf{V}^{[j,i]} = \left[\mathbf{v}_1^{[j,i]}\ \mathbf{v}_2^{[j,i]}\ \cdots \mathbf{v}_d^{[j,i]}\right]$ and $\mathbf{s}^{[j,i]}$ denotes the d data symbol

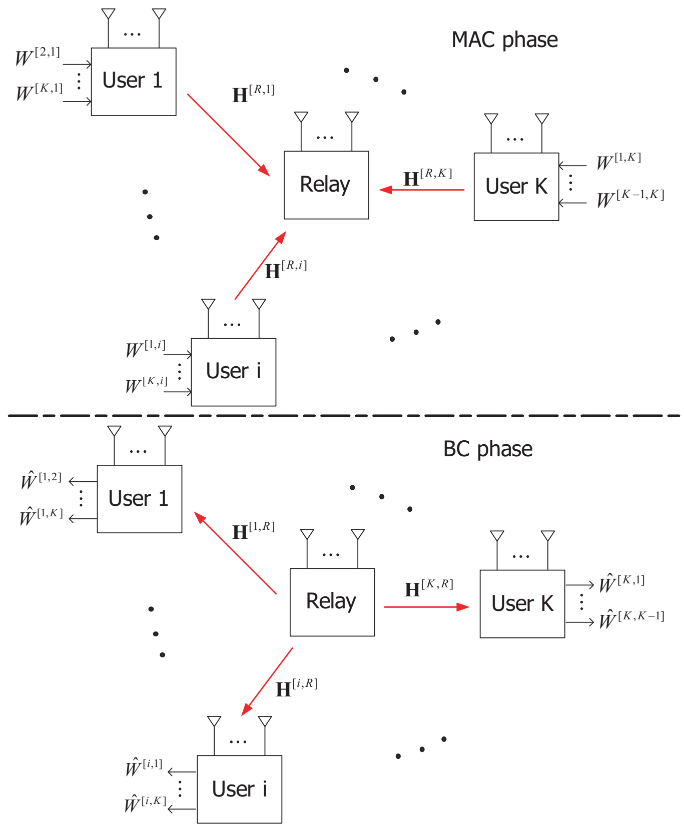

Fig. 1. System model of K user MIMO Y channels.

vector $\mathbf{s}^{[j,i]} = \left[s_1^{[j,i]}\ s_2^{[j,i]}\ \cdots\ s_d^{[j,i]}\right]^T$. We assume that each data symbol has unit variance and each user has the average power constraint $\mathbb{E}\left\{\mathrm{tr}\left[\mathbf{x}^{[i]}\mathbf{x}^{[i]^H}\right]\right\} \leq P$.

The received signal at the relay is obtained by

$$\mathbf{y}^{[R]} = \sum_{i=1}^{K} \mathbf{H}^{[R,i]}\mathbf{x}^{[i]} + \mathbf{n}^{[R]}, \tag{1}$$

where $\mathbf{H}^{[R,i]}$ stands for the $N \times M_i$ channel matrix from user i to the relay, and $\mathbf{n}^{[R]}$ indicates the additive white Gaussian noise (AWGN) vector with zero mean and unit variance. After receiving the signal $\mathbf{y}^{[R]}$, the relay generates the new transmit signal $\mathbf{x}^{[R]}$ as

$$\mathbf{x}^{[R]} = \gamma \mathbf{F}\mathbf{y}^{[R]}$$

where $\mathbf{F}$ equals the beamforming matrix of size $N \times N$, and $\gamma = \sqrt{P/\mathbb{E}\left\{\mathrm{tr}\left[\mathbf{F}\mathbf{y}^{[R]}\mathbf{y}^{[R]^H}\mathbf{F}^H\right]\right\}}$ is the power normalizing factor to satisfy the relay power constraint $\mathbb{E}\left\{\mathrm{tr}\left[\mathbf{x}^{[R]}\mathbf{x}^{[R]^H}\right]\right\} \leq P$.

In the BC phase where the relay broadcasts $\mathbf{x}^{[R]}$ to all users, the received signal vector at user j is written as

$$\mathbf{y}^{[j]} = \mathbf{H}^{[j,R]}\mathbf{x}^{[R]} + \mathbf{n}^{[j]},$$

where $\mathbf{H}^{[j,R]}$ denotes the $M_j \times N$ channel matrix from the relay to user j and $\mathbf{n}^{[j]}$ represents the AWGN vector with zero mean and unit variance at user j. User j applies the receive combining matrix $\mathbf{W}^{[j,i]} = \left[\mathbf{w}_1^{[j,i]}\ \mathbf{w}_2^{[j,i]}\ \cdots\ \mathbf{w}_d^{[j,i]}\right]$ to decode the user i's message $W^{[j,i]}$.

Then, the $d \times 1$ receive filter output vector for the message $W^{[j,i]}$ is computed as

$$\begin{aligned}\hat{\mathbf{y}}^{[j,i]} &= \mathbf{W}^{[j,i]^H}\mathbf{y}^{[j]} \\ &= \mathbf{W}^{[j,i]^H}\Big(\mathbf{H}^{[j,R]}\gamma\mathbf{F}\mathbf{y}^{[R]} + \mathbf{n}^{[j]}\Big) \\ &= \gamma\mathbf{W}^{[j,i]^H}\mathbf{H}^{[j,R]}\mathbf{F}\mathbf{H}^{[R,i]}\mathbf{V}^{[j,i]}\mathbf{s}^{[j,i]} + \tilde{\mathbf{a}}^{[j,i]} + \tilde{\mathbf{b}}^{[j,i]} + \tilde{\mathbf{n}}^{[j,i]},\end{aligned} \tag{2}$$

where $\tilde{\mathbf{a}}^{[j,i]} = \gamma\mathbf{W}^{[j,i]^H}\mathbf{H}^{[j,R]}\mathbf{F}\sum_{k=1,k\neq i,j}^{K}\mathbf{H}^{[R,i]}\mathbf{x}^{[k,i]}$, $\tilde{\mathbf{b}}^{[j,i]} = \gamma\mathbf{W}^{[j,i]H}\mathbf{H}^{[j,R]}\mathbf{F}\sum_{m=1,m\neq i}^{K}\sum_{k=1}^{K}\mathbf{H}^{[R,m]}\mathbf{x}^{[k,m]}$ and $\tilde{\mathbf{n}}^{[j,i]} = \gamma\mathbf{W}^{[j,i]H}\mathbf{H}^{[j,R]}\mathbf{F}\mathbf{n}^{[R]} + \mathbf{W}^{[j,i]}\mathbf{n}^{[j]}$ are intra-user interference, inter-user interference and noise terms at user j, respectively. Also, the self-interference in the transmitted signal is eliminated in (3), since each terminal node knows its own transmitted data.

Throughout this paper, it is assumed that all users and the relay operate in the full-duplex mode. This implies that all terminals can receive and transmit simultaneously. We assume generic channel matrices where all channel elements are generated from an independently and identically distributed complex Gaussian distribution with zero mean and unit variance. Also, the channel is assumed to be known perfectly at the transmitters and receivers. The SNR is defined as P. Because implementation of multi-user detection may be difficult, we assume that the interference terms are treated as noise. Then, for a given set of the beamforming matrices $\mathbf{V}^{[i,j]}$ and $\mathbf{W}^{[j,i]}$ for $\forall i,j$, the achievable rate for $W^{[j,i]}$ is obtained as

$$\begin{aligned}R^{[j,i]}(P) &= \log_2\det\left(\mathbb{E}\left\{\hat{\mathbf{y}}^{[j,i]}\hat{\mathbf{y}}^{[j,i]^H}\right\}\right) \\ &-\log_2\det\left(\mathbb{E}\left\{\tilde{\mathbf{a}}^{[j,i]}\tilde{\mathbf{a}}^{[j,i]^H} + \tilde{\mathbf{b}}^{[j,i]}\tilde{\mathbf{b}}^{[j,i]^H} + \tilde{\mathbf{n}}^{[j,i]}\tilde{\mathbf{n}}^{[j,i]^H}\right\}\right).\end{aligned} \tag{3}$$

Then, a sum of the achievable DOF for K-user the MIMO Y channels is denoted as

$$\eta_{sum}(K) = \lim_{P\to\infty}\sum_{i=1}^{K}\sum_{j=1,j\neq i}^{K}\frac{R^{[j,i]}(P)}{\log_2 P}.$$

Also, we define the normalized DOF which is normalized by the average number of antennas of all nodes as

$$\bar{\eta}(K) = \frac{\text{the sum of the achievable DOF}}{\text{the average number of node antennas}}. \tag{4}$$

This can be interpreted as a measurement of the antenna efficiency for communication in the interference channel environment.

III. Achievable Degrees of Freedom of K-user MIMO Y Channels

In this section, we investigate the achievable DOF and the normalized DOF of K-user MIMO Y channels. To this end, we first consider the symmetric case where every user and the relay have an equal number of antennas. Subsequently for the case of an unequal number of antennas, we propose a new scheme which requires a smaller number of antennas compared to the case of equal number of antennas and study the feasibility condition for this scheme.

A. Case of the equal number of antennas

We first consider the symmetric antenna case where all users and the relay have the equal number of antennas, i.e., $M_i = N = M$ for $\forall i$. The number of required antennas to achieve the DOF of $dK(K-1)$ is presented in the following theorem.

Theorem 1: For the symmetric case of K-user Y channels where all nodes have M antennas and each user wishes to obtain the DOF of d for each message, the total DOF $\eta_{sum} = dK(K-1)$ is achieved if

$$M \geq \frac{dK(K-1)}{2}. \tag{5}$$

Proof: We show the achievability using signal space alignment for network coding and the AF relaying strategy. In the MAC phase, user i sends $d(K-1)$ independent data streams along with the $M \times d$ transmit beamforming matrices $\mathbf{V}^{[j,i]}$ for $j \in \{1,2,\ldots,K\}/\{i\}$. Since every user simultaneously sends $d(K-1)$ independent data streams, the relay receives $dK(K-1)$ competing flows of information. In the subsequent BC phase, the relay forwards the received $dK(K-1)$ data streams to all users. If each user has $M \geq dK(K-1)$ antennas, the user is able to get $d(K-1)$ desired data symbols by eliminating the other users' $d(K-1)^2$ interfering data symbols. However, if all nodes have antennas less than the number of total data streams $(i.e., M < dK(K-1))$, for example $M = \frac{dK(K-1)}{2}$, the user cannot resolve the $d(K-1)$ desired data symbols using overall M dimensional space.

In order to overcome this problem due to lack of dimensionality, two users who wish to exchange messages cooperatively design the beamforming matrices $\mathbf{V}^{[j,i]}$ and $\mathbf{V}^{[i,j]}$ so that two effective MAC channels $\mathbf{H}^{[R,i]}\mathbf{V}^{[j,i]}$ and $\mathbf{H}^{[R,j]}\mathbf{V}^{[i,j]}$ are aligned at the receiver of the relay. More specifically, the transmit beamforming vectors are determined such that the following conditions are satisfied

$$\text{span}\left(\mathbf{H}^{[R,i]}\mathbf{v}_m^{[j,i]}\right) \doteq \text{span}\left(\mathbf{H}^{[R,j]}\mathbf{v}_m^{[i,j]}\right), \tag{6}$$

for $m = 1,\cdots,d$, where $\text{span}(\mathbf{A}) \doteq \text{span}(\mathbf{B})$ indicates that the subspaces spanned by two matrices $\mathbf{A}$ and $\mathbf{B}$ are equivalent.

Since (6) is equivalent to $\text{span}\left(\mathbf{v}_m^{[j,i]}\right) \doteq \text{span}\left(\mathbf{H}^{[R,i]^{-1}}\mathbf{H}^{[R,j]}\mathbf{v}_m^{[i,j]}\right)$, we can easily design $\mathbf{V}^{[j,i]}$ and $\mathbf{V}^{[i,j]}$ by choosing any d vectors among the eigenvectors of the matrix $\mathbf{H}^{[R,i]^{-1}}\mathbf{H}^{[R,j]}$. Then, two desired signal vectors of user i and user j are aligned for the m-th data stream. The relay is able to detect two signals on this aligned space and to encode them by the network coding methods [4][5]. This scheme is referred to as SSA-NC [13]. It is worth to note that the SSA-NC allows the relay to exploit the signal dimension efficiently by processing the $dK(K-1)$ data streams in the reduced $\frac{dK(K-1)}{2}$ dimensional space with network coding.

After obtaining $\frac{dK(K-1)}{2}$ network coded data symbols, the relay just forwards the received signals to all users using an arbitrary beamfoming scheme. Since the user has $M \geq \frac{dK(K-1)}{2}$ antennas, $\frac{dK(K-1)}{2}$ signals can be estimated by employing a zero-forcing receiver. Each user picks the desired $d(K-1)$ signals among the detected $\frac{dK(K-1)}{2}$ signals and removes the $d(K-1)$ self-interference signals. As a result, each user can get its desired $d(K-1)$ data streams. ■

For K-user Y channels where all nodes have M antennas which achieve the DOF of $dK(K-1)$, the normalized DOF in (4) can be calculated using Theorem 1 as

$$\bar{\eta}^{(equal)}(K) \quad = \quad \frac{dK(K-1)}{M} \leq 2.$$

In this channel, at most the normalized DOF of 2 is achievable for any number of users K. For two user case which is a two-way relay channel, it is well known that the normalized DOF is 2 and is achieved by using the network coding and self-interference cancellation. This is referred as the *network coding gain* which is defined as the ratio of the number of transmissions required by the non-coding approach to the minimum number of transmissions used by network coding to exchange messages [19]. We can observe that the K-user Y channel also obtains the network coding gain as in two-way relay channels.

A similar concept of the normalized DOF was considered in [20] and [21] for the case of K-user interference channels with constant channel coefficients, which is normalized by the single-user's DOF in the absence of interference, and we indicate it as $\hat{\eta}(K)$. For a symmetric system $(M \times N, d)^K$ denoting the K-user MIMO interference channel where every transmitter has M antennas, every receiver has N antennas and each user wishes to achieve the DOF of d, the authors in [20] showed that $\hat{\eta}(K)$ is bounded by

$$\hat{\eta}(K) = \frac{dK}{\min(M,N)} \leq 1 + \frac{\max(M,N)}{\min(M,N)} - \frac{d}{\min(M,N)}.$$

For the special case of the symmetric square case ($M=N$), it is proved in [21] that $\hat{\eta}(K)$ satisfies $\frac{dK}{M} = \frac{K}{M}\lfloor\frac{2M}{K+1}\rfloor \leq 2\frac{K}{K+1}$. Then, we see that the normalized DOF $\hat{\eta}(K)$ of 2 is achievable at most, which accounts for *alignment gain* [21]. In the context of a newly defined DOF which is normalized by the average number of node antennas, $\bar{\eta}(K)$ for $(M \times N, d)^K$ systems is upper-bounded as

$$\bar{\eta}(K) = \frac{dK}{(M+N)/2} \leq 2\frac{K}{K+1}$$

by using Theorem 1 in [20] and we can check that an alignment gain of 2 is obtained as $K \to \infty$.

This normalized DOF $\bar{\eta}(K)$ motivates us to explore a scheme which can achieve both network coding gain and alignment gain for K-user Y channels. For the case of the equal number of antennas, we do not get the alignment gain since we have enough antennas for detecting messages and do not need to consider alignment for interference during the BC phase. We can improve the normalized DOF by using antennas more efficiently with the reduced number of users' antennas. In the next subsection, we will show how we obtain not only the network coding gain but also the alignment gain.

B. Case of the unequal number of antennas

Here, we consider the asymmetric case where the number of antennas for the users and the relay is unequal. Our question is how many antennas are required in minimum to achieve the DOF of $dK(K-1)$ for K-user MIMO Y channels. To answer this, we will propose a new method using SSA-NC for both the MAC and BC phase and the AF relaying method. Also we analyze the feasibility condition for the proposed scheme. We present the main result of this section in the following theorem.

Theorem 2: If $N = \frac{dK(K-1)}{2}$, $\min\{M_i + M_j, \ \forall\ i \neq j\} \geq N + d$ and $M_i \geq d(K-1)$ for all $i \in \{1, 2, \ldots, K\}$, then the DOF of $dK(K-1)$ is achieved.

Proof: The proof of the theorem involves three steps: the MAC phase, the BC phase and the relaying beamforming matrix design. We will describe each step sequentially.

1) MAC phase: Each user sends d independent messages to the other $K-1$ users using the SSA-NC scheme. The condition of the beamforming vector designs for SSA-NC between user i and user j is represented in (6). For designing the transmit beamforming vectors $\mathbf{v}_m^{[j,i]}$ and $\mathbf{v}_m^{[i,j]}$ satisfying the condition (6), the direction vectors $\tilde{\mathbf{v}}_m^{[j,i]}$ and $\tilde{\mathbf{v}}_m^{[i,j]}$ can be obtained by solving the following linear equation

$$\begin{bmatrix} \mathbf{I}_N & -\mathbf{H}^{[R,i]} & \mathbf{0} \\ \mathbf{I}_N & \mathbf{0} & -\mathbf{H}^{[R,j]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_m^{\pi(i,j)} \\ \tilde{\mathbf{v}}_m^{[j,i]} \\ \tilde{\mathbf{v}}_m^{[i,j]} \end{bmatrix} = \mathbf{0}, \qquad (7)$$

where $\mathbf{u}_m^{\pi(i,j)}$ indicates the m-th basis vector in the intersection subspace between user i and user j with the constraint $\|\mathbf{u}_m^{\pi(i,j)}\|^2 = 1$ and $\pi(i,j)$ represents an index for the group of user i and user j. Note that the vectors $\mathbf{u}_m^{\pi(i,j)}$ for $m = 1, \cdots, d$ should be linearly independent in order to obtain d network coded messages.

To show the existence of a solution of equation (7) for $m = 1, \cdots, d$, we should check the dimension of the nullspace of the matrix in (7) whose size is $2N \times (N + M_i + M_j)$. Recall that all channel elements are independently drawn from a continuous distribution. Therefore, if $M_i + M_j \geq N + d$, a solution for SSA-NC exists with probability one, since the rank of the nullpace of the matrix in (7) is equal to or greater than d. From the above SSA-NC conditions, $dK(K-1)$ independent messages can be contained within the $\frac{dK(K-1)}{2}$ dimensional signal space. Therefore, it is enough for the relay to have $N = \frac{dK(K-1)}{2}$ antennas for conveying $dK(K-1)$ messages. Then, we have the conditions $N = \frac{dK(K-1)}{2}$, $M_i + M_j \geq N + d$ for all $i, j \in \{1, 2, \ldots, K\}, i \neq j$, and $M_i \geq d(K-1)$. Note that the last condition $M_i \geq d(K-1)$ naturally comes from the fact that each user transmits $d(K-1)$ independent messages.

As a result, the received signal at the relay in (1) can be rewritten as

$$\begin{aligned} \mathbf{y}^{[R]} &= \sum_{i=1}^{K} \mathbf{H}^{[R,i]} \sum_{j=1, j\neq i}^{K} \sum_{m=1}^{d} \mathbf{v}_m^{[j,i]} s_m^{[j,i]} + \mathbf{n}^{[R]} \\ &= \mathbf{U}^{[R]} \mathbf{s}^{[R]} + \mathbf{n}^{[R]}, \end{aligned} \qquad (8)$$

where $\mathbf{U}^{[R]}$ and $\mathbf{s}^{[R]}$ are defined by $\mathbf{U}^{[R]} = \left[\mathbf{u}_1^{\pi(1,2)} \cdots \mathbf{u}_d^{\pi(1,2)} \mathbf{u}_1^{\pi(1,3)} \cdots \mathbf{u}_d^{\pi(K,K-1)}\right]$ and $\mathbf{s}^{[R]} = \left[s_1^{\pi(1,2)} \cdots s_d^{\pi(1,2)} s_1^{\pi(1,3)} \cdots s_d^{\pi(K,K-1)}\right]^T$, respectively, and $s_m^{\pi(i,j)} = \alpha_m^{[i,j]} s_m^{[i,j]} + \alpha_m^{[j,i]} s_m^{[j,i]}$ is the superimposed signal with the power normalizing factors $\alpha_m^{[j,i]}$ and $\alpha_m^{[i,j]}$. Here,

$\alpha_m^{[j,i]}$ is determined to make $\mathbf{v}_m^{[j,i]} = \alpha_m^{[j,i]}\tilde{\mathbf{v}}_m^{[j,i]}$ satisfy the average power constraint at user i. Note that $\mathbf{U}^{[R]}$ has a size of $N \times \frac{dK(K-1)}{2}$ with $N = \frac{dK(K-1)}{2}$ and all columns are linearly independent with probability one by the assumption that all channel matrices are generic. Thus, the relay is able to obtain $\frac{dK(K-1)}{2}$ network coded messages from $\mathbf{y}^{[R]}$ by nulling the interference vectors.

2) BC phase: Now, we will show how we transmit these network coded messages to the corresponding users. In a similar way as in the MAC phase, we can also align the user pair's received signal space. We first design the receive combining matrices $\mathbf{W}^{[i,j]}$ and $\mathbf{W}^{[j,i]}$ for decoding the message $W^{[i,j]}$ and $W^{[j,i]}$. They should satisfy the following condition as

$$\text{span}\left(\mathbf{w}_m^{[i,j]H}\mathbf{H}^{[i,R]}\right) \doteq \text{span}\left(\mathbf{w}_m^{[j,i]H}\mathbf{H}^{[j,R]}\right),$$

for $m = 1, \cdots, d$. Thus, these receive combining vectors are obtained by solving the equation

$$\left[\begin{array}{ccc} \mathbf{z}_m^{\pi(i,j)^H} & \mathbf{w}_m^{[i,j]^H} & \mathbf{w}_m^{[j,i]^H} \end{array}\right]\left[\begin{array}{cc} \mathbf{I}_N & \mathbf{I}_N \\ -\mathbf{H}^{[j,R]} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H}^{[i,R]} \end{array}\right] = \mathbf{0}, \quad (9)$$

where $\mathbf{z}_m^{\pi(i,j)}$ stands for the m-th basis vector in the intersection subspace between user i and user j with the constraint $\|\mathbf{z}_m^{\pi(i,j)}\|^2 = 1$.

Then, $\mathbf{z}_m^{\pi(i,j)}$, $\mathbf{w}_m^{[j,i]}$ and $\mathbf{w}_m^{[i,j]}$ are determined by the left nullspace of the $(N + M_i + M_j) \times 2N$ matrix in (9) for $m = 1, \cdots, d$. Since $M_i + M_j \geq N + d$, the dimension of the left nullspace is at least d. The receive filter output vector in (2) is rewritten as

$$\hat{\mathbf{y}}^{[j]} = \left[\begin{array}{cccc} \mathbf{Z}^{\pi(j,1)} & \mathbf{Z}^{\pi(j,2)} & \cdots & \mathbf{Z}^{\pi(j,K)} \end{array}\right]^H \mathbf{x}^{[R]} + \mathbf{W}^{[j]H}\mathbf{n}^{[j]}$$

where $\mathbf{Z}^{\pi(j,i)} = \left[\mathbf{z}_1^{\pi(j,i)}\mathbf{z}_2^{\pi(j,i)}\cdots\mathbf{z}_d^{\pi(j,i)}\right]$ and $\mathbf{W}^{[j]} = \left[\mathbf{W}^{[j,1]}\mathbf{W}^{[j,2]}\cdots\mathbf{W}^{[j,K]}\right]$.

Suppose that all elements of $\mathbf{H}^{[j,R]}$ and $\mathbf{H}^{[i,R]}$ are randomly generated from a continuous distribution and thus $\mathbf{z}_m^{\pi(i,j)}$ for $\forall i,\ j,\ m$ can be found to satisfy the condition (9). Then, $\frac{dK(K-1)}{2}$ vectors $\mathbf{z}_m^{\pi(i,j)}$ for $\forall i,\ j,\ m$ are linearly independent with probability one. We have to transmit the network coded message $s_m^{\pi(i,j)}$ so that it arrives at this aligned column space $\mathbf{z}_m^{\pi(i,j)}$ without any interference. From the relay perspective, this system is equivalent to a multiple input single output (MISO) broadcast channel where the base station has $\frac{dK(K-1)}{2}$ antennas and each $\frac{dK(K-1)}{2}$ user has a single antenna. Also, the total effective channel matrix $\mathbf{Z}^{[R]} = \left[\mathbf{Z}^{\pi(1,2)}\ \mathbf{Z}^{\pi(1,3)}\cdots\ \mathbf{Z}^{\pi(K-1,K)}\right]$ is composed of these aligned vectors. Therefore, it can simply cancel all interference by using zero-forcing at the relay.

Let us denote $\tilde{\mathbf{Z}}_m^{\pi(i,j)}$ as the matrix which excludes the m-th aligned signal space vector of the index $\pi(i,j)$, $\mathbf{z}_m^{\pi(i,j)}$, from the matrix $\mathbf{Z}^{[R]}$. For example, we have $\tilde{\mathbf{Z}}_2^{\pi(1,3)} = \left[\mathbf{Z}^{\pi(1,2)}\ \mathbf{z}_1^{\pi(1,3)}\mathbf{z}_3^{\pi(1,3)}\cdots\ \mathbf{z}_d^{\pi(1,3)}\mathbf{Z}^{\pi(1,4)}\cdots\mathbf{Z}^{\pi(K-1,K)}\right]$. Then for the network coded message $s_m^{\pi(i,j)}$, the relay beamforming vector $\mathbf{v}_m^{\pi(i,j)}$ belongs to the nullspace of $\tilde{\mathbf{Z}}_m^{\pi(i,j)^H}$, i.e., $\mathbf{v}_m^{\pi(i,j)} \in \text{null}\left(\tilde{\mathbf{Z}}_m^{\pi(i,j)^H}\right)$. Since $\tilde{\mathbf{Z}}_m^{\pi(i,j)}$ is a $N \times \left(\frac{dK(K-1)}{2} - 1\right)$ matrix, the precoding vector $\mathbf{v}_m^{\pi(i,j)}$ always exists as long as $\frac{dK(K-1)}{2} - 1 < N$. Finally, we can transmit $s_m^{\pi(i,j)}$ to user i and user j without interfering other users' messages.

3) Relay filter design: Given $\mathbf{U}^{[R]}$ and $\mathbf{Z}^{[R]}$, we compute the relay beamforming matrix $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{V}^{[R]}\mathbf{W}^{[R]H} \quad (10)$$

where $\mathbf{V}^{[R]} = \left[\mathbf{v}_1^{\pi(1,2)}\mathbf{v}_2^{\pi(1,2)}\cdots\mathbf{v}_d^{\pi(K,K-1)}\right]$ and $\mathbf{W}^{[R]} = \left[\mathbf{w}_1^{\pi(1,2)}\mathbf{w}_2^{\pi(1,2)}\cdots\mathbf{w}_d^{\pi(K,K-1)}\right]$ represent the transmit beamforming and the receive combining matrix, respectively. Here $\mathbf{v}_m^{\pi(i,j)}$ and $\mathbf{w}_m^{\pi(i,j)}$ are designed by $\mathbf{v}_m^{\pi(i,j)} \in \text{null}\left(\tilde{\mathbf{Z}}_m^{\pi(i,j)^H}\right)$ and $\mathbf{w}_m^{\pi(i,j)} \in \text{null}\left(\tilde{\mathbf{U}}_m^{\pi(i,j)^H}\right)$, respectively. Here, $\tilde{\mathbf{U}}_m^{\pi(i,j)^H}$ denotes the matrix which excludes the m-th vector of $\pi(i,j)$, $\mathbf{u}_m^{\pi(i,j)}$, from the matrix $\mathbf{U}^{[R]}$.

Then, the received signals of user i can be written as

$$\begin{aligned} \hat{\mathbf{y}}^{[i]} &= \mathbf{W}^{[i]H}\mathbf{H}^{[i,R]}\mathbf{x}^{[R]} + \mathbf{W}^{[i]H}\mathbf{n}^{[i]} \\ &= \mathbf{W}^{[i]H}\mathbf{H}^{[i,R]}\gamma\mathbf{F}\left(\mathbf{U}^{[R]}\mathbf{s}^{[R]} + \mathbf{n}^{[R]}\right) + \mathbf{W}^{[i]H}\mathbf{n}^{[i]} \\ &= \gamma\mathbf{W}^{[i]H}\mathbf{H}^{[i,R]}\mathbf{V}^{[R]}\mathbf{s}^{[R]} + \gamma\mathbf{W}^{[i]H}\mathbf{H}^{[i,R]}\mathbf{F}\mathbf{n}^{[R]} + \mathbf{W}^{[i]H}\mathbf{n}^{[i]} \\ &= \gamma\mathbf{H}_{eff}^{[i,R]}\mathbf{s}^{[i,R]} + \gamma\mathbf{W}^{[i]H}\mathbf{H}^{[i,R]}\mathbf{F}\mathbf{n}^{[R]} + \mathbf{W}^{[i]H}\mathbf{n}^{[i]}, \quad (11) \end{aligned}$$

where $\mathbf{s}^{[i,R]}$ is user i's only desired symbol vector among $\mathbf{s}^{[R]}$, i.e., $\mathbf{s}^{[i,R]} = \left[s_1^{\pi(i,1)}s_2^{\pi(i,1)}\cdots s_d^{\pi(i,K)}\right]^T$ and $\mathbf{H}_{eff}^{[i,R]}$ denotes the effective channel matrix as $\mathbf{H}_{eff}^{[i,R]} = \text{diag}\left\{g_1^{\pi(i,1)}, g_2^{\pi(i,1)}, \cdots, g_d^{\pi(i,K)}\right\}$ with $g_m^{\pi(i,j)} = \mathbf{z}_m^{\pi(i,j)^H}\mathbf{v}_m^{\pi(i,j)}\mathbf{w}_m^{\pi(i,j)^H}\mathbf{u}_m^{\pi(i,j)}$ being the effective channel gain of the corresponding symbol $s_m^{\pi(i,j)}$.

After removing the self-interference $s_m^{[j,i]}$, user i can detect user j's transmitted signals with the effective SNR as

$$\text{SNR}_m^{[i,j]} = \frac{\gamma|g_m^{\pi(i,j)}|^2\alpha^{[i,j]2}}{\|\gamma\mathbf{w}_m^{[j,i]H}\mathbf{H}^{[i,R]}\mathbf{F}\|^2 + \|\gamma\mathbf{w}_m^{[j,i]H}\|^2}.$$

Then, the achievable rate for $W^{[i,j]}$ in (3) is simply calculated stream-wisely as $R^{[i,j]}(P) = \sum_{m=1}^{d}\log_2(1 + \text{SNR}_m^{[i,j]})$. Compared with (3), note that the inter-user interferences $\tilde{\mathbf{a}}^{[i,j]}$ and $\tilde{\mathbf{b}}^{[i,j]}$ can be perfectly removed, and thus we can transmit $dK(K-1)$ symbols without any interference. As a result, the DOF of $dK(K-1)$ can be achieved by SSA-NC during both the MAC and BC phase if $N = \frac{dK(K-1)}{2}$, $\min\{M_i + M_j,\ \forall\, i \neq j\} \geq N + d$ and $M_i \geq d(K-1)$ for all $i \in \{1, 2, \ldots, K\}$. ■

We conclude this section with the following remarks.

Remark 1: The superimposed messages $\mathbf{s}^{[R]}$ in (8) can be regenerated by ANC [4] or PNC [5]. For example, the PNC creates the XORed message $W^{\pi(i,j)} = W^{[i,j]} \oplus W^{[j,i]}$ by allocating the power to make both $\alpha_m^{[j,i]}$ and $\alpha_m^{[i,j]}$ equal, while the ANC yields $\mathbf{s}^{[R]}$ itself in the case of $\alpha_m^{[j,i]} \neq \alpha_m^{[i,j]}$. In this paper, we consider the ANC for examining the sum rate of the channel under the power constraint.

Remark 2: In [13], during the BC phase, the network coding based zero-forcing (ZF) scheme was proposed, which is optimal in terms of the DOF for $K = 3$. This scheme,

TABLE I
REQUIRED NUMBER OF ANTENNAS OF SSA-NC FOR K-USER MIMO Y CHANNELS ($d = 1$)

K	Achievable DOF	N	M_1	M_2	M_3	M_4	M_5
3	6	3	2	2	2	·	·
4	12	6	3	4	4	4	·
5	20	10	5	6	6	6	6

however, has a drawback when K is larger than 3, because more relay antennas are required to satisfy the zero inter-user interference condition as K increases. For example in [13], the case of $K = 4$ is considered with $N = 7$, $M_i = 4$ for $\forall i$ and $d = 1$ where the DOF of $dK(K-1) = 12$ is achieved. Here, the relay constructed the beamforming vectors using the ZF method so that the uninteresting signals are eliminated at the unintended receiver of the user. For instance, $s^{\pi(1,2)} = \alpha^{[2,1]}s^{[2,1]}+\alpha^{[1,2]}s^{[1,2]}$ is an intended symbol to user 1 and user 2, but it acts as an interference to user 3 and user 4. Therefore, $\mathbf{v}^{\pi(1,2)}$ is designed to be aligned in the intersection subspace between both nullspace of $\mathbf{H}^{[3,R]}$ and $\mathbf{H}^{[4,R]}$ as

$$\mathbf{v}^{\pi(1,2)} \in \left\{\text{null}(\mathbf{H}^{[3,R]}) \cap \text{null}(\mathbf{H}^{[4,R]})\right\}. \tag{12}$$

However, this solution of $\mathbf{v}^{\pi(1,2)}$ only exists when $N > M_1 + M_2$. Thus, the relay adopts an antenna selection scheme in order to resolve the dimensionality constraint of (12). By choosing $\hat{M} = 3$ antennas among total $M = 4$ antennas, the size of the effective channel matrices $\hat{\mathbf{H}}^{[j,R]}$ from the relay to user j becomes the 3×7. This makes it possible to compute the beamforming vector $\mathbf{v}^{\pi(1,2)}$ which satisfies the ZF constraint

$$\begin{bmatrix} \hat{\mathbf{H}}^{[3,R]} \\ \hat{\mathbf{H}}^{[4,R]} \end{bmatrix} \mathbf{v}^{\pi(1,2)} = \mathbf{0}.$$

Compared with this ZF method, our proposed scheme can reduce the number of relay antennas from 7 to 6 by applying SSA-NC during the BC phase.

Remark 3: Using these feasibility conditions, we can provide some criteria for multi-user multi-directional MIMO communication systems. Table I shows the required number of antennas of SSA-NC for K-user MIMO Y channels which satisfy the feasibility condition with $d = 1$. We can see that as the number of users K increases, the number of the relay antennas and user antennas grows in the order of $\frac{K^2}{2}$ and $\frac{K^2}{4}$, respectively.

Remark 4: For the unequal number of antenna case, the normalized DOF is upper-bounded by

$$\begin{aligned} \bar{\eta}^{(unequal)}(K) &= \frac{dK(K-1)}{(\sum_{i=1}^{K} M_i + N)/(K+1)} \\ &\leq 4\frac{K-1}{K}, \end{aligned}$$

where the equality holds by setting $N = \frac{dK(K-1)}{2}$, $M_i = \lceil \frac{N+d}{2} \rceil, \forall i$ which satisfy the feasibility condition in Theorem 2 and $d = 4$. As $K \to \infty$, we can see that the normalized DOF of 4 is achievable. This means that the proposed scheme obtains not only a *network coding gain* of 2 but also an *alignment gain* of 2.

IV. ACHIEVABLE DEGREES OF FREEDOM OF K-USER SISO Y CHANNELS

In this section, we consider K-user single input single output (SISO) Y channels where all nodes have a single antenna. We will show that for this channel, the total DOF of 2 is achieved by having $d = \frac{2}{K(K-1)}$ for every $K(K-1)$ messages. This comes from Theorem 1 which implies $d \leq \frac{2M}{K(K-1)}$. This result will be derived by using the rational dimension framework introduced in [17], [18] and [22] for K user interference channels and the multihop interference channels with constant channel coefficients.

In fact, the DOF of 2 can also be achieved by letting only two users active at each time, i.e., two-way relay channel. Then, time sharing among different user pairs can attain the same DOF as rational dimension alignment. However, the motivation behind using the rational dimension framework here is that it does not need time sharing. This means that if we apply a time division multiple access method, the data delay will be increased as K grows, because each user pair has to wait until the time they are allowed to use spectrum. However, we can show that each user pair in the proposed method can get the DOF of $\frac{2}{K(K-1)}$ without making a significant delay.

We assume that all signals, noise and channel coefficients are real values and the result can be generalized to the complex case by Theorem 7 in [23]. For a single antenna system, the channel outputs for the MAC and BC phase are represented as

$$y^{[R]} = \sum_{j=1}^{K} h^{[R,j]}x^{[j]} + n^{[R]}$$

and

$$y^{[j]} = h^{[j,R]}x^{[R]} + n^{[j]}, \quad \text{for } j = 1, \cdots, K$$

where $h^{[R,j]}$ and $h^{[j,R]}$ indicate the channel coefficient from the user j to the relay and from the relay to the user j for $j \in \{1, \cdots, K\}$, the channel input $x^{[j]}$ is the transmitted signal with an average power constraint as $\mathbb{E}\left\{{x^{[j]}}^2\right\} \leq P$ at user j, and $n^{[j]}$ denotes the real valued AWGN with zero mean and unit variance.

In this real channel case, the DOF for the message from user i to user j is defined as [18]

$$\eta^{[j,i]} = \lim_{P\to\infty} \frac{R^{[j,i]}(P)}{\frac{1}{2}\log_2 P}.$$

Now we will show that the total DOF of 2 with $d = \frac{2}{K(K-1)}$ is achievable when all nodes have a single antenna. For simple representation, we first prove the 3 user case whose result can be easily generalized to the K user case.

1) Sources: At each source node, user i for $i = 1, 2, 3$ wants to transfer two messages $W^{[j,i]}$ for $j \in \{1, 2, 3\}$ and $j \neq i$. The message $W^{[j,i]}$ is encoded to transmit data symbols $s^{[j,i]}$. We assume that all data symbols use the same constellation $\mathcal{C}_S$ with integer points from interval $[-Q, Q]$ with $Q = \lfloor \gamma P^{\frac{1-\epsilon}{2(3+\epsilon)}} \rfloor$ where γ and $\epsilon < 1$ are two arbitrary constants, i.e., $\mathcal{C}_S = [-Q, Q]_{\mathbb{Z}} = \{-Q, -Q+1, \ldots, Q-1, Q\}$. Then, the data symbol $s^{[j,i]}$ is uniformly selected on $\mathcal{C}_S$.

The transmitted signal at user i is the linear combination of two integer data symbols $s^{[j,i]}$ for $j \in \{1,2,3\}/\{i\}$ with real coefficients $v^{[j,i]}$, which can be written as

$$x^{[i]} = A \sum_{j=1, j\neq i}^{3} v^{[j,i]} s^{[j,i]}$$

where A is a normalizing factor for satisfying the transmit power constraint. The transmit power of user i is constrained as

$$E\left[{x^{[i]}}^2\right] = A^2 \sum_{j=1,j\neq i}^{3} {v^{[j,i]}}^2 E\left[{s^{[j,i]}}^2\right] \leq A^2 {\eta^{[i]}}^2 \gamma^2 P^{\frac{1-\epsilon}{(3+\epsilon)}} \leq P$$

where ${\eta^{[i]}}^2 = \sum_{j=1,j\neq i}^{3} {v^{[j,i]}}^2$ for $i = 1,2,3$. In order to satisfy the power constraints at all source nodes, we set A as

$$A = \frac{\eta_S}{\gamma} P^{\frac{2+2\epsilon}{2(3+\epsilon)}} \tag{13}$$

where $\eta_S = \min\left\{\frac{1}{\eta^{[1]}}, \frac{1}{\eta^{[2]}}, \frac{1}{\eta^{[3]}}\right\}$.

Now, we choose real coefficients $v^{[j,i]}$ to achieve the alignment conditions at the relay as

$$\begin{aligned} h^{[R,1]} v^{[2,1]} &= h^{[R,2]} v^{[1,2]}, \\ h^{[R,2]} v^{[3,2]} &= h^{[R,3]} v^{[2,3]}, \\ h^{[R,3]} v^{[1,3]} &= h^{[R,1]} v^{[3,1]}. \end{aligned} \tag{14}$$

2) Relay: The received signal at the relay becomes

$$\begin{aligned} y^{[R]} &= h^{[R,1]}x^{[1]} + h^{[R,2]}x^{[2]} + h^{[R,3]}x^{[3]} + n^{[R]} \\ &= Ah^{[R,1]} \sum_{j=1,j\neq 1}^{3} v^{[j,1]} s^{[j,1]} + Ah^{[R,2]} \sum_{j=1,j\neq 2}^{3} v^{[j,2]} s^{[j,2]} \\ &\quad + Ah^{[R,3]} \sum_{j=1,j\neq 3}^{3} v^{[j,3]} s^{[j,3]} + n^{[R]} \\ &= Ah^{[R,1]} v^{[2,1]} (s^{[2,1]} + s^{[1,2]}) + Ah^{[R,2]} v^{[3,2]} (s^{[3,2]} + s^{[2,3]}) \\ &\quad + Ah^{[R,3]} v^{[1,3]} (s^{[1,3]} + s^{[3,1]}) + n^{[R]} \end{aligned}$$

where the last equality comes from the alignment condition (14).

Let us denote $s_1^{[R]} = s^{[2,1]} + s^{[1,2]}$, $s_2^{[R]} = s^{[3,2]} + s^{[2,3]}$ and $s_3^{[R]} = s^{[1,3]} + s^{[3,1]}$. Note that since $s_i^{[R]}$ $(i = 1,2,3)$ are the sum of two integer symbols, they are also integer numbers which are elements in a new integer set $[-2Q, 2Q]_{\mathbb{Z}}$. Therefore, the received signal at the relay $y^{[R]}$ is a point of the following constellation $\mathcal{C}_R$ with a noise.

$$\mathcal{C}_R = \left\{ A\left(h^{[R,1]} v^{[2,1]} s_1^{[R]} + h^{[R,2]} v^{[3,2]} s_2^{[R]} + h^{[R,3]} v^{[1,3]} s_3^{[R]} \right) \right\}.$$

Unlike an AF relay method as in the linear scheme, the relay makes hard decisions on the signals received in each rational dimension. Note that $v^{[2,1]}$, $v^{[3,2]}$ and $v^{[1,3]}$ are rationally independent almost surely for the generic channel coefficients and thus the relay is able to detect the point in $\mathcal{C}_R$ which has the minimum distance. From the key results using Khintchine-Groshev theorem in [17] and [18], the minimum distance between points in the received constellation $\mathcal{C}_R$ is given by

$$d_{\min}(\mathcal{C}_R) > \frac{\kappa A}{(2Q)^{2+\epsilon}}, \tag{15}$$

where κ is an arbitrary constant.

In $\mathcal{C}_R$, the relay performs a hard decision on $s_i^{[R]}$ by mapping the point to $\hat{s}_i^{[R]}$ for $i = 1,2,3$. Note that the error probability of the hard decision for $s_i^{[R]}$, $\Pr(\hat{s}_i^{[R]} \neq s_i^{[R]})$, can be approximated as

$$\Pr(\hat{s}_i^{[R]} \neq s_i^{[R]}) \leq \mathbf{Q}\left(\frac{d_{\min}}{2}\right) \leq \exp\left(-\frac{d_{\min}^2}{8}\right), \tag{16}$$

where $\mathbf{Q}(x)$ is a Q-function and is upper bounded as $\mathbf{Q}(x) \leq \exp\left(-\frac{x^2}{2}\right)$. By substituting (13) and (15) in (16), it can be bounded as

$$\Pr(\hat{s}_i^{[R]}) \neq s_i^{[R]}) \leq \exp\left(-\delta P^{\epsilon}\right) \tag{17}$$

where δ is a constant which is a function of γ, κ and η_S. Thus, it can be shown that the error probability $\Pr(\hat{s}_i^{[R]} \neq s_i^{[R]})$ diminishes as the power P goes to infinity.

Using these estimated data symbols, the relay creates the transmit signal for the destination nodes. The transmit signal at the relay R is expressed as

$$x^{[R]} = B \sum_{j=1}^{3} v_j^{[R]} \hat{s}_j^{[R]}$$

where $v_j^{[R]}$ is a real coefficient for the data symbol $\hat{s}_j^{[R]}$ and B indicates a normalizing constant to satisfy the power constraint. Since $\hat{s}_j^{[R]} \leq 2Q$ and the transmit power of the relay R is represented as

$$E\left[{x^{[R]}}^2\right] = B^2 \sum_{j=1}^{3} {v_j^{[R]}}^2 E\left[{\hat{s}_j^{[R]}}^2\right] \leq 4B^2 {\eta^{[R]}}^2 \gamma^2 P^{\frac{1-\epsilon}{(3+\epsilon)}} \leq P$$

where $\eta^{[R]} = \sum_{j=1}^{3} {v_j^{[R]}}^2$, we set B as

$$B = \frac{1}{2\eta^{[R]}\gamma} P^{\frac{2+2\epsilon}{2(3+\epsilon)}}.$$

3) Destinations: Let us assume that the relay perfectly estimates $\hat{s}_j^{[R]}$ for $\forall j$, i.e., $\hat{s}_j^{[R]} = s_j^{[R]}$. This assumption becomes true as $P \to \infty$. Then, the received signal at user 1 is

$$\begin{aligned} y^{[1]} &= h^{[1,R]} x^{[R]} + n^{[1]} \\ &= Bh^{[1,R]} \left(v_1^{[R]} s_1^{[R]} + v_2^{[R]} s_2^{[R]} + v_3^{[R]} s_3^{[R]} \right) + n^{[1]}. \end{aligned} \tag{18}$$

The desired symbols $s^{[1,2]}$ and $s^{[1,3]}$ are contained in $s_1^{[R]} = s^{[2,1]} + s^{[1,2]}$ and $s_3^{[R]} = s^{[1,3]} + s^{[3,1]}$, respectively, and they can be decoded after self-interference cancellation using its transmitted data symbol $s^{[2,1]}$ and $s^{[3,1]}$. Then, (18) becomes

$$y^{[1]} = Bh^{[1,R]} \left(v_1^{[R]} \hat{s}^{[1,2]} + v_3^{[R]} \hat{s}^{[1,3]} + v_2^{[R]} (s^{[3,2]} + s^{[2,3]}) \right) + n^{[1]}.$$

Similar to the detection at the relay, it can be shown from [17] that the error probability of estimation for $\hat{s}^{[1,2]}$ and $\hat{s}^{[1,3]}$ approaches zero as $P \to \infty$ for any arbitrary rational numbers $v_1^{[R]}$, $v_2^{[R]}$ and $v_3^{[R]}$.

Now, we can derive a lower bound of the achievable rate for the messages $W^{[1,2]}$ and $W^{[1,3]}$ using the result in [17].

Each message $W^{[1,k]}$ is able to achieve a rate as

$$\begin{aligned} I(s^{[1,k]};\hat{s}^{[1,k]}) &= H(s^{[1,k]}) - H(s^{[1,k]}|\hat{s}^{[1,k]}) \\ &\geq H(s^{[1,k]}) - \left(1+\Pr(\hat{s}^{[1,k]}\neq s^{[1,k]})\log_2|\mathcal{C}_S|\right) \\ &\geq \log_2|\mathcal{C}_S| - 1 - \Pr(\hat{s}^{[1,k]}\neq s^{[1,k]})\log_2|\mathcal{C}_S| \\ &= \left(1-\Pr(\hat{s}^{[1,k]}\neq s^{[1,k]})\right)\log_2|\mathcal{C}_S| - 1, \quad (19) \end{aligned}$$

for $k = 2, 3$, where the second and the third inequality comes from Fano's inequality and the fact that $\mathcal{C}_S$ has the uniform distribution, respectively. Note that $\log_2|\mathcal{C}_S|$ can be represented as $\log_2|\mathcal{C}_S| = \frac{1-\epsilon}{2(3+\epsilon)}\log_2 P + o(\log_2 P)$. Since $\Pr(\hat{s}^{[1,k]} \neq s^{[1,k]})$ will go to zero as $P \to \infty$, an achievable rate for $W^{[1,k]}$ in (19) becomes $\frac{1-\epsilon}{2(3+\epsilon)}\log_2 P + o(\log_2 P)$ and therefore the DOF of $\frac{1-\epsilon}{3+\epsilon}$ is obtained, i.e., $\eta^{[1,k]} = \frac{1-\epsilon}{3+\epsilon}$.

In the same way, the received signals at user 2 and user 3 become

$$y^{[2]} = Bh^{[2,R]}\left(v_2^{[R]}\hat{s}^{[2,3]} + v_1^{[R]}\hat{s}^{[2,1]} + v_3^{[R]}(s^{[3,1]} + s^{[1,3]})\right) + n^{[2]},$$
$$y^{[3]} = Bh^{[3,R]}\left(v_3^{[R]}\hat{s}^{[3,1]} + v_2^{[R]}\hat{s}^{[3,2]} + v_1^{[R]}(s^{[1,2]} + s^{[2,1]})\right) + n^{[3]}.$$

By using the same argument for user 1, we can show that the same DOF of $\frac{1-\epsilon}{3+\epsilon}$ is achieved for $\forall\eta^{[j,i]}$. Therefore, each messages obtains $d = \frac{1-\epsilon}{3+\epsilon}$ and the achievable DOF becomes $\sum_{k=1}^{3}\sum_{j=1,j\neq k}^{3}\eta^{[k,j]} = 6\frac{1-\epsilon}{3+\epsilon}$. Since ϵ can be made arbitrarily small, we can make the DOF arbitrarily close to 2 for the Y channel when all nodes have a single antenna.

Remark 5: In [24], the authors derived upper-bounds for the sum-capacity of 3 user SISO Y channels. The pre-log factor of this upper-bound becomes 1 in the real channel as SNR increases, and this can be generalized to the complex case with the DOF of 2. In addition, they showed the achievable lower-bounds which maintain the constant gap between sum-capacity upper and lower bounds. It is achieved by the "functional decode and forward (DF)" scheme where time is divided into frames of 3 time slots, and only 2 users and the relay are operating in each slot. Constraining this symbol extension to a single time slot for completing one communication cycle, we can show that the DOF of 2 is obtained by using the rational dimension framework.

Remark 6: In this section, we have considered the 3 user case and it can be generalized to the K-user case. We change the constellation size as $\mathcal{C}_S = [-Q, Q]_{\mathbb{Z}}$ with $Q = \lfloor \gamma P^{\frac{1-\epsilon}{2(K(K-1)/2+\epsilon)}} \rfloor$. Real value beamforming coefficients $v^{[j,i]}$ and $v^{[i,j]}$ are chosen in order to meet the alignment conditions as $h^{[r,i]}v^{[j,i]} = h^{[r,j]}v^{[i,j]}$ for $\forall i \neq j$ and the corresponding power factor A are multiplied for satisfying the power constraint. Then, we can prove that each data achieves the DOF of $\frac{1-\epsilon}{K(K-1)/2+\epsilon}$ in the same way as the 3 user case. Finally, the total achievable DOF is $\sum_{k=1}^{K}\sum_{j=1,j\neq k}^{K}\eta^{[k,j]} = \frac{K(K-1)(1-\epsilon)}{K(K-1)/2+\epsilon}$, and with arbitrarily small ϵ, K-user Y channels also obtain the DOF of 2. Note that the normalized DOF in (4) is $\bar{\eta}(K) = 2$ and thus the K-user SISO Y channel only achieves the network coding gain.

V. Simulation Results

In this section, we provide the sum rate performance of the proposed scheme through simulations. From the simulation results, we will verify the feasibility conditions in Theorem 2. For the sake of notational conveniences, we denote $\{N|(M_1, M_2, \ldots, M_K)|d\}$ for K-user MIMO Y channels where the relay has N antennas and user i employs M_i transmit antennas and wish to achieve the DOF of d for each message. It is assumed that each user allocates equal power to each stream, i.e., $\|\mathbf{v}_m^{[j,i]}\|^2 = \frac{P}{d(K-1)}$ for $\forall i, j, m$.

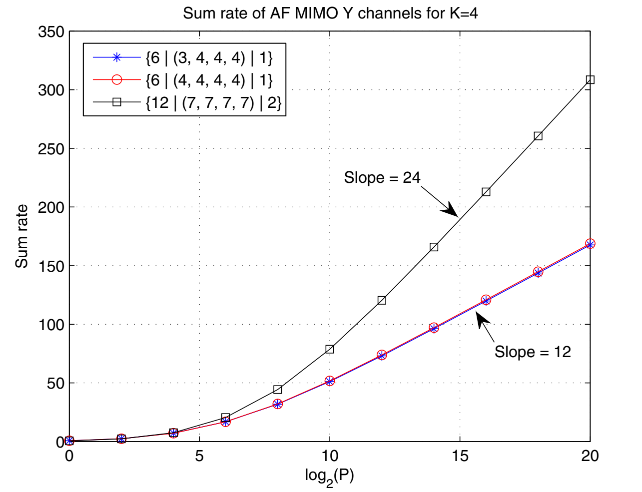

Fig. 2. Sum rate performance of various antenna configurations for $K = 4$.

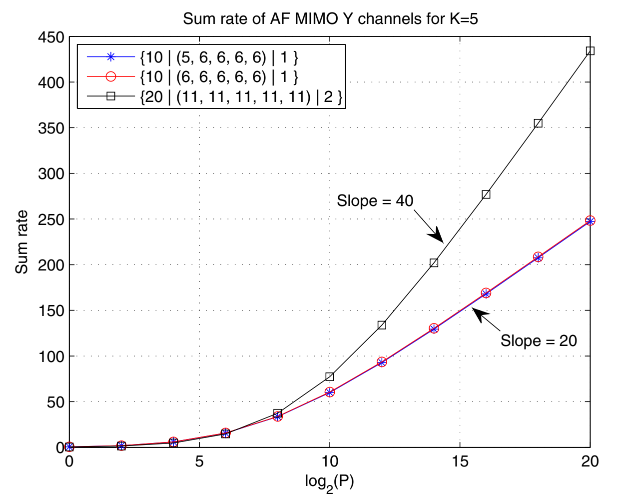

Fig. 3. Sum rate performance of various antenna configurations for $K = 5$.

Figures 2 and 3 illustrate the sum rate performance of $K = 4$ and 5, respectively, with various configurations in terms of $\log_2 P$. The slope of the sum rate curves equals the achievable DOF. For both the symmetric user antenna case $\{6|(4,4,4,4)|1\}$ and the asymmetric user antenna case $\{6|(3,4,4,4)|1\}$, the DOF of 12 is achieved with $K = 4$ when the feasibility condition is satisfied. Compared with $d = 1$, for the case of $d = 2$, the number of relay antenna should be twice. However, the required antenna for each user is less than twice in order to achieve the DOF of 24, i.e., $\{12|(7,7,7,7)|2\}$ as shown in Fig. 2. We also obtain the DOF of 20 and 40 for the 5 user case when $M_i \geq 4d$, $N \geq 10d$, and $N < \min\{M_i + M_j - d, \ \forall i \neq j\}$. We confirm that the achievable

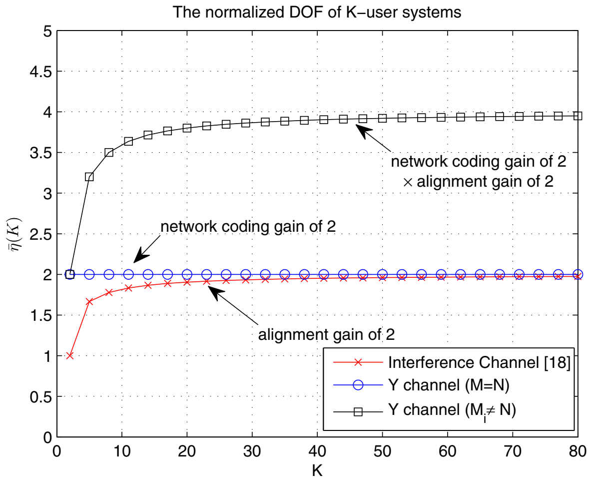

Fig. 4. The DOF normalized by the average number of node antennas.

DOF is 20 for both the symmetric case $\{10|(6,6,6,6,6)|1\}$ and the asymmetric case $\{10|(5,6,6,6,6)|1\}$, and is 40 for $\{20|(11,11,11,11,11)|2\}$ as shown in Fig. 3.

Fig. 4 exhibits the DOF normalized by the average number of node antennas, $\bar{\eta}(K)$, for various channel models with respect to the number of users. We assume that all antenna configurations satisfy the feasibility condition for achieving the total DOF. Then, as the number of users increases, the normalized DOF of K-user interference channels with constant channel coefficients becomes 2, which accounts for an alignment gain. For the symmetric case of K-user Y channels ($M_i = N = M, \forall i$), the normalized DOF of 2 is achieved regardless of the number of users, due to a network coding gain. Finally, for the asymmetric case ($M_i \neq N, \forall i$), we can reduce the ratio of users' antennas and relay antennas by using the proposed SSA-NC scheme as the number of users increases. As a result, the normalized DOF becomes $\lim_{K\to\infty} \bar{\eta}(K) = 4$ and it is shown that the proposed scheme obtains both an alignment gain of 2 and a network coding gain of 2.

VI. Conclusion

In this paper, we have studied K-user Y channels where K users exchange messages with each other through an intermediate relay. We have shown that for multiple antenna Y channels, we can attain the DOF of $dK(K-1)$ by using the signal space alignment for network coding during both the BC and MAC phase. We have investigated the feasibility condition on the required number of antennas for each user and relay. Also we show that this proposed scheme obtains both an alignment gain of 2 and a network coding gain of 2 in terms of the normalized DOF. For single antenna Y channels, we have verified that the DOF of 2 is achieved for any user numbers using the rational dimension framework. Our derived condition is confirmed for various configurations and this serves as a lower bound of the capacity for K-user MIMO Y channels. Optimal beamforming vector designs of SSA-NC and power allocation issues will remain as future works.

References

[1] C. E. Shannon, "Two-way communication channels," in *Proc. 1961 Berkeley Symp. Math. Stat. Prob.*, vol. 1, pp. 611–644.

[2] B. Rankov and A. Wittneben, "Spectral efficient protocols for half-duplex fading relay channels," *IEEE J. Sel. Areas Commun.*, vol. 25, pp. 379–389, Feb. 2007.

[3] K.-J. Lee, H. Sung, E. Park, and I. Lee, "Joint optimization for one and two-way MIMO AF multiple-relay systems," *IEEE Trans. Wireless Commun.*, vol. 9, pp. 3671–3681, Dec. 2010.

[4] S. Katti, S. Gollakota, and D. Katabi, "Embracing wireless interference: analog network coding," in *Proc. 2007 SIGCOMM*.

[5] S. Zhang and S.-C. Liew, "Channel coding and decoding in a relay system operated with physical-layer network coding," *IEEE J. Sel. Areas Commun.*, vol. 27, pp. 788–796, June 2009.

[6] M. Chen and A. Yener, "Multiuser two-way relaying for interference limited systems," in *Proc. 2008 IEEE ICC*.

[7] S. Avestimehr, A. Khajehnejad, A. Sezgin, and B. Hassibi, "Capacity region of the deterministic multi-pair bi-directional relay network," in arXiv:0903.2653, Mar. 2009.

[8] A. Sezgin, M. A. Khajehnejad, A. S. Avestimehr, and B. Hassibi, "Approximate capacity region of the two-pair bidirectional Gaussian relay network," in *Proc. 2009 IEEE International Symp. Inf. Theory*.

[9] K. Lee, S.-H. Park, J.-S. Kim, and I. Lee, "Degrees of freedom on MIMO multi-link two-way relay channels," in *Proc. 2010 IEEE Globecom*.

[10] N. Lee and J. Chun, "Signal space alignment for an encryption message and successive network code decoding on the MIMO K-way relay channel," in *Proc. 2011 IEEE ICC*.

[11] D. Gunduz, A. Yener, A. Goldsmith, and H. V. Poor, "The multi-way relay channel," in *Proc. 2009 IEEE International Symp. Inf. Theory*.

[12] N. Lee and J.-B. Lim, "A novel signaling for communication on MIMO Y channel: signal space alignment for network coding," in *Proc. 2009 International Symp. Inf. Theory*.

[13] N. Lee, J.-B. Lim, and J. Chun, "Degrees of the freedom of the MIMO Y channel: signal space alignment for network coding," *IEEE Trans. Inf. Theory*, vol. 56, pp. 3332–3342, July 2010.

[14] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: interference alignment, decomposition, and performance analysis," *IEEE Trans. Inf. Theory*, vol. 54, pp. 3457–3470, Aug. 2008.

[15] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the K-User interference channel," *IEEE Trans. Inf. Theory*, vol. 54, pp. 3425–3441, Aug. 2008.

[16] S.-H. Park and I. Lee, "Degrees of freedom of multiple broadcast channels in the presence of inter-cell interference," *IEEE Trans. Commun.*, vol. 59, pp. 1481–1487, May 2011.

[17] A. S. Motahari, S. O. Gharan, and A. K. Khandani, "Real interference alignment with real numbers," in arXiv:0908.1208, Aug. 2009.

[18] A. S. Motahari, S. O. Gharan, M.-A. Maddah-Ali, and A. K. Khandani, "Real interference alignment: exploiting the potential of single antenna systems," in arXiv:0908.2282, Nov. 2009.

[19] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Medard, and J. Crowcroft, "XORs in the air: practical wireless network coding," *IEEE/ACM Trans. Netw.*, vol. 16, pp. 497–510, June 2008.

[20] C. M. Yetis, T. Gou, S. A. Jafar, and A. H. Kayran, "On feasibility of interference alignment in MIMO interference networks," *IEEE Trans. Signal Process.*, vol. 58, pp. 4771–4782, Sep. 2010.

[21] G. Bresler, D. Cartwright, and D. Tse, "Settling the feasibility of interference alignment for the MIMO interference channel: the symmetric square case," in arXiv:1104.0888, Apr. 2011.

[22] T. Gou, S. A. Jafar, S.-W. Jeon, and S.-Y. Chung, "Aligned interference neutralization and the degrees of freedom of the 2x2x2 interference channel," in arXiv:1012.2350, Dec. 2010.

[23] M. A. Maddah-Ali, "On the degrees of freedom of the compound MIMO broadcast channels with finite states," in arXiv:0909.5006, Oct. 2009.

[24] A. Chaaban, A. Sezgin, and S. Avestimehr, "On the sum capacity of the Y-channel," in arXiv:1102.2787, Feb. 2011.

Kwangwon Lee (S'10) received the B.S. and M.S. degrees in electrical engineering from Korea University, Seoul, Korea in 2006 and 2008, respectively. He is currently working toward the Ph.D. degree at Korea University, Seoul, Korea. During the spring in 2009, he visited University of Southern California, Los Angeles, CA, to conduct collaborative research under the Brain Korea 21 (BK21) Program. He was awarded the Bronze Prize in the 2007 Samsung Humantech Paper Contest in February 2008. His research interests are communication theory and signal processing techniques for multi-user multi-way wireless networks using interference management and network coding.

Namyoon Lee received the B.S. degree in radio and communication engineering from Korea University, Seoul, Korea, in 2006 and the M.S. degree in electrical engineering from the Korea Advanced Institute of Science and Technology (KAIST), Daejeon, Korea, in 2008. From 2008 to 2011, he was a member of technical staff at Samsung Advanced Institute of Technology (SAIT) and Samsung Electronics Co. Ltd. in Korea, where he investigated next generation device-to-device (D2D) wireless communication systems and involved standardization activities of the 3GPP LTE-Adv., especially for femto-cell deployment. He is currently a Ph.D. candidate at the University of Texas at Austin. His current research interests are multiuser and multiway-communication theory using interference alignment, neutralization, and network coding.

Mr. Lee was a recipient of the 2009 Samsung Best Paper Award. He was also awarded several fellowships, including the Graduate Student Research Fellowship from the Korea Science and Engineering Foundation (KOSEF) in 2006; the Korea Government Fellowship from 2006 to 2007; the Kwanjeong Educational Foundation Fellowship in 2011.

Inkyu Lee (S'92-M'95-SM'01) received the B.S. degree (Hon.) in control and instrumentation engineering from Seoul National University, Seoul, Korea, in 1990, and the M.S. and Ph.D. degrees in electrical engineering from Stanford University, Stanford, CA, in 1992 and 1995, respectively. From 1995 to 2001, he was a Member of Technical Staff at Bell Laboratories, Lucent Technologies, where he conducted research on high-speed wireless system designs. He later worked for Agere Systems (formerly Microelectronics Group of Lucent Technologies), Murray Hill, NJ, as a Distinguished Member of Technical Staff from 2001 to 2002. In September 2002, he joined the faculty of Korea University, Seoul, Korea, where he is currently a Professor in the School of Electrical Engineering. During 2009, he visited University of Southern California, LA, USA, as a visiting Professor. He has published around 70 journal papers in IEEE, and has 30 U.S. patents granted or pending. His research interests include digital communications and signal processing techniques applied for next generation wireless systems. Dr. Lee currently serves as an Associate Editor for IEEE TRANSACTIONS ON COMMUNICATIONS and the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. Also, he has been a Chief Guest Editor for the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS (Special Issue on 4G Wireless Systems). He received the IT Young Engineer Award as the IEEE/IEEK joint award in 2006, and received the Best Paper Award at APCC in 2006 and IEEE VTC in 2009. Also he was a recipient of the Hae-Dong Best Research Award of the Korea Information and Communications Society (KICS) in 2011.