MMSE-Based MIMO Cooperative Relaying Systems: Closed-Form Designs and Outage Behavior

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***Abstract*—In this paper, we investigate minimum mean squared error (MMSE) based amplify-and-forward cooperative multiple antenna relaying systems where a non-negligible direct link exists between the source and the destination. First, we provide a new design strategy for optimizing the relay amplifying matrix. Instead of conventional optimal design approaches resorting to an iterative gradient method, we propose a near optimal closed-form solution which provides an insight. As relay systems with a direct link incur a non-convex problem in general, we exploit the decomposable property of the error covariance matrix and a relaxation technique imposing a structural constraint on the problem. Next, we study the error performance limit of the proposed scheme using diversity-multiplexing tradeoff analysis, which leads to several interesting observations on MMSE-based cooperative relaying systems. Finally, through numerical simulations, we confirm that the proposed solution shows the performance very close to the optimum with much reduced complexity and the analysis closely matches with simulation results.**

***Index Terms*—MIMO, MMSE, Relay, Direct link, Closed-form design, DMT analysis.**

I. Introduction

IN A RECENT decade, it has been well recognized that multiple-input and multiple-output (MIMO) wireless systems can improve link performance and spectral efficiency by utilizing diversity and multiplexing gains [1]–[3]. Recently, relay cooperative techniques have also garnered a significant interest thanks to the advantages such as extended cell coverage and improved reliability [4]–[9]. Because of these benefits, MIMO relaying systems have been considered as a powerful candidate for next generation wireless networks [10]–[12].

In practical relay networks, one of the most popular relaying protocols is amplify-and-forward (AF) due to its simplicity, which amplifies the signal received from the source and forward it to the destination [7]–[9]. In AF MIMO relaying systems, designs of the optimum amplifying matrix (or transceiver) at the relay have been active research areas over the past few years. In *pure relaying* channels which do not have a direct link from the source to the destination, an analysis based on information theoretic approaches has been reported in [13] and [14]. Running parallel with this, for reducing the decoding complexity, minimum mean squared error (MMSE) based methods have also been investigated for the relay matrix design in the literature [15]–[18]. However, in *cooperative relaying* channels where there exists a non-negligible direct link, all these works become suboptimal.

In fact, a source-to-destination direct link can provide a valuable multiplexing gain as well as a diversity gain [8]. Hence, if the direct link is available, we need to optimize the relay matrix considering the direct link. For a single stream transmission, this problem is simply convex, because the direct link has no influence on the relay filter design [19] [20]. On the contrary, in case of multiple spatial streams, the direct link incurs non-convexity of the problem which is much more challenging. Recently, the authors in [21] attempted to find the optimal solution with respect to the MMSE criterion resorting to an iterative method such as a gradient descent algorithm. Although the optimal scheme in [21] successfully minimizes the mean squared error (MSE), the iterative method hardly provides helpful insights.

Regarding MMSE-based cooperative relaying systems, this paper contains two main contributions: a closed-form relay matrix design and its asymptotic performance analysis. The first part of the paper introduces a new design strategy of the relay transceiver with a non-iterative manner. Instead of conventional canonical coordination methods [15] [21], we address the problem with an error decomposition approach which was first found in [16] over pure relaying channels. More specifically, utilizing the fact that the MMSE optimal relay matrix can generally be expressed as a combination of the relay receiver and the precoder, we prove that the error covariance matrix at the destination can be represented as a sum of two individual covariance matrices, which allows us to considerably simplify the problem and to obtain valuable insights on MIMO relaying systems. Note that our result cannot be derived from [16] and the proof is even more challenging, because the relay receiver and the precoder with a direct link do not follow simple Wiener filter structures. Also, in the presence of a direct link, the decomposed problem is still non-convex. Therefore, we employ a relaxation technique imposing a structural constraint on the relay precoder to

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identify an insightful closed-form solution. For this reason, our solution does not ensure optimality, but numerical results demonstrate that we can achieve the performance very close to the optimal design [21] with much reduced complexity.

In the second part of the paper, we present the diversity-multiplexing tradeoff (DMT) analysis of the proposed scheme, which provides a compact characterization of the tradeoff between the transmission rate and the diversity order [2], and gives a convenient tool for comparing the proposed scheme with various relaying systems with different protocols [8]–[11]. Due to difficulties in finding an exact DMT expression, we first establish an upperbound of the DMT which offers a theoretical limit of the system and derive an achievable DMT of the proposed scheme. Then, it is shown that our achievable DMT coincides with the upperbound in many cases with practical antenna configurations. This result illustrates the optimality of our solution in terms of the outage behavior and provides a helpful guideline for designing MMSE-based relaying systems. One interesting observation made from our analysis is that in a specific antenna configuration, increasing the number of relay antennas does not provide any performance advantage. Note that in MMSE-based cooperative relaying systems, there is no reported work for the analytical performance that can explain our observations. Finally, computer simulations show that the numerical performance of the optimal scheme [21] as well as the proposed scheme is accurately predicted by our achievable DMT expression.

The rest of the paper is organized as follows: Section II provides the signal model for cooperative AF MIMO relaying systems. In Section III, we formulate the MMSE problem. Then, we propose a near optimal closed-form solution in Section IV and the DMT is analyzed in Section V. Section VI illustrates simulation results. Finally, our conclusions are drawn in Section VII.

Throughout this paper, normal letters represent scalar quantities, boldface letters indicate vectors and boldface uppercase letters designate matrices. We use $\mathbb{S}_+^N$ to denote a set of $N\times N$ positive semi-definite matrices. The superscripts $(\cdot)^T$, $(\cdot)^H$, and $(\cdot)^*$ stand for transpose, conjugate transpose, and element-wise conjugate, respectively. $\mathbf{I}_N$ is defined as an $N\times N$ identity matrix and $E[\cdot]$ denotes the expectation operator. $\mathrm{Tr}\,(\mathbf{A})$ (or $\mathrm{tr}\,(\mathbf{A})$) and $[\mathbf{A}]_{k,k}$ indicate the trace and the k-th diagonal element of a matrix $\mathbf{A}$, respectively.

II. System Model

In this paper, we consider a cooperative relaying system in Fig. 1 where one AF relay node helps communication between the source and the destination in the presence of a direct link. The source, relay, and destination nodes are equipped with N_t, N_r, and N_d antennas, respectively. As in conventional relay optimization strategies [13]–[16], we assume that no channel state information (CSI) is allowed at the source, while both the relay and the destination have perfect CSI of all links[1]. Due to loop interference in the relay node, it is assumed that each data transmission occurs in two separate time slots.

[1]It is also possible to extend our algorithm to source-relay joint design scheme with full CSI at the source, but we defer this to the future work.

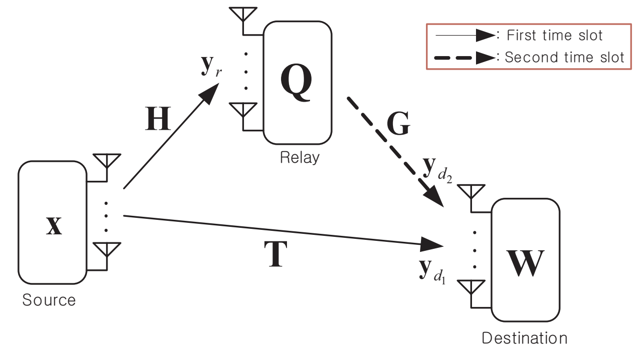

Fig. 1. System description of a MIMO cooperative AF relay network

In the first time slot, the source broadcasts the signal vector $\mathbf{x}\in\mathbb{C}^{N_t\times 1}$ with power constraint $E[\|\mathbf{x}\|^2]\le P_T$ to both the relay and the destination, and the received signals at the relay and at the destination, $\mathbf{y}_r\in\mathbb{C}^{N_r\times 1}$ and $\mathbf{y}_{d_1}\in\mathbb{C}^{N_d\times 1}$, are respectively given by

$$\mathbf{y}_r=\mathbf{H}\mathbf{x}+\mathbf{n}_r \quad \text{and} \quad \mathbf{y}_{d_1}=\mathbf{T}\mathbf{x}+\mathbf{n}_{d_1},$$

where $\mathbf{H}\in\mathbb{C}^{N_r\times N_t}$ and $\mathbf{T}\in\mathbb{C}^{N_d\times N_t}$ denote the source-to-relay and the source-to-destination (direct link) channel matrices, respectively, and $\mathbf{n}_r\in\mathbb{C}^{N_r\times 1}$ and $\mathbf{n}_{d_1}\in\mathbb{C}^{N_d\times 1}$ indicate the noise vectors at the relay and at the destination, respectively.

Next, in the subsequent time slot, the relay signal $\mathbf{y}_r$ is precoded by the relay transceiver $\mathbf{Q}\in\mathbb{C}^{N_r\times N_r}$ and transmitted to the destination. Then, the received signal at the destination is written by

$$\mathbf{y}_{d_2}=\mathbf{G}\mathbf{Q}\mathbf{H}\mathbf{x}+\mathbf{n}_{d_2},$$

where $\mathbf{n}_{d_2}\triangleq\mathbf{G}\mathbf{Q}\mathbf{n}_r+\mathbf{n}_d$ designates the effective noise vector in the second time slot with covariance matrix $\mathbf{R}_n\triangleq\mathbf{G}\mathbf{Q}\mathbf{Q}^H\mathbf{G}^H+\mathbf{I}_{N_d}$. In this case, the relay matrix $\mathbf{Q}$ needs to satisfy the relay power constraint P_R as $E[\|\mathbf{Q}\mathbf{y}_r\|^2]\le P_R$. We assume that all channel matrices have random entries which are independent and identically distributed complex Gaussian, i.e., $\sim\mathcal{CN}(0,\sigma_h^2)$, $\sim\mathcal{CN}(0,\sigma_g^2)$, and $\sim\mathcal{CN}(0,\sigma_t^2)$ for $\mathbf{H}$, $\mathbf{G}$, and $\mathbf{T}$, respectively, but remain constant over a codeword duration (quasi-static Rayleigh fading). Here, the terms σ_h^2, σ_g^2, and σ_t^2 reflect the pathloss effect in each link. All elements of noise vectors $\mathbf{n}_r$, $\mathbf{n}_d$ and $\mathbf{n}_{d_1}$ are also assumed to be spatially and temporally white Gaussian with zero mean unit variance $\sim\mathcal{CN}(0,1)$.

As a result, combining two signals received at the destination over two consecutive time slots, we have the signal vector $\mathbf{y}_d\in\mathbb{C}^{2N_d\times 1}$ at the destination as

$$\mathbf{y}_d=\begin{bmatrix}\mathbf{y}_{d_1}\\ \mathbf{y}_{d_2}\end{bmatrix}=\begin{bmatrix}\mathbf{T}\\ \mathbf{G}\mathbf{Q}\mathbf{H}\end{bmatrix}\mathbf{x}+\begin{bmatrix}\mathbf{n}_{d_1}\\ \mathbf{n}_{d_2}\end{bmatrix}. \tag{1}$$

Finally, when a linear receiver $\mathbf{W}\in\mathbb{C}^{N_t\times 2N_d}$ is employed at the destination, the estimated signal waveform $\mathbf{s}\in\mathbb{C}^{N_t\times 1}$ is expressed as $\mathbf{s}=\mathbf{W}\mathbf{y}_d$. Note that the rank of the effective channel in (1) is always smaller than or equal to N_d+N where $N\triangleq\min(N_r,N_d)$. As the focus of this paper is the MMSE spatial equalizer design, we make a fundamental assumption that the number of spatial streams N_t is constrained by the effective channel rank as $N_t\le N_d+N$. Note that only $N_t\le$

N has been assumed for pure relaying systems with no direct link [15]–[18].

III. Problem Formulation

In this section, we formulate a problem optimizing the relay transceiver $\mathbf{Q}$ under the MMSE criterion. We first derive the error covariance matrix as a function of $\mathbf{Q}$, and then show that from the MMSE point of view, it can be represented as a sum of two individual covariance matrices. Using this property, we can substantially simplify the problem.

Defining the error vector as $\mathbf{e} \triangleq \mathbf{s} - \mathbf{x} = \mathbf{W}\mathbf{y}_d - \mathbf{x}$, the joint optimization problem for minimizing the MSE is mathematically expressed as

$$\min_{\mathbf{W},\mathbf{Q}} \mathrm{Tr}\left(\mathbf{R}_e(\mathbf{W},\mathbf{Q})\right) \quad s.t.\ \ \mathrm{Tr}\left(\mathbf{Q}(\rho\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})\mathbf{Q}^H\right) \leq P_R, \tag{2}$$

where $\mathbf{R}_e(\mathbf{W},\mathbf{Q}) \triangleq E[\mathbf{e}\mathbf{e}^H]$ denotes the error covariance matrix as a function of $\mathbf{W}$ and $\mathbf{Q}$, and $\rho \triangleq P_T/N_t$ indicates the input signal-to-noise ratio (SNR). It is easy to verify that this problem is convex (or quasi-convex) with respect to each of $\mathbf{W}$ and $\mathbf{Q}$, although it is generally non-convex in the joint optimization perspective.

Therefore, for given $\mathbf{Q}$, the optimum receive filter $\hat{\mathbf{W}}$ is simply obtained as [22]

$$\hat{\mathbf{W}}(\mathbf{Q}) = \left(\mathbf{H}^H\mathbf{Q}^H\mathbf{G}^H\mathbf{R}_n^{-1}\mathbf{G}\mathbf{Q}\mathbf{H} + \mathbf{T}^H\mathbf{T} + \rho^{-1}\mathbf{I}_{N_t}\right)^{-1}\mathbf{H}_S^H \tag{3}$$

where $\mathbf{H}_S \triangleq \begin{bmatrix}\mathbf{T}^T & (\mathbf{R}_n^{-1}\mathbf{G}\mathbf{Q}\mathbf{H})^T\end{bmatrix}^T$. Then, substituting $\hat{\mathbf{W}}(\mathbf{Q})$ into $\mathbf{R}_e(\mathbf{W},\mathbf{Q})$ and invoking the matrix inversion lemma [16], it can be rephrased as a function of $\mathbf{Q}$ as

$$\mathbf{R}_e(\mathbf{Q}) = \left(\mathbf{H}^H\mathbf{Q}^H\mathbf{G}^H\mathbf{R}_n^{-1}\mathbf{G}\mathbf{Q}\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1}, \tag{4}$$

where $\mathbf{R}_T \triangleq (\mathbf{T}^H\mathbf{T} + \rho^{-1}\mathbf{I}_{N_t})^{-1}$.

Now, we look at the problem (2) in terms of the relay matrix $\mathbf{Q}$. Once $\mathbf{W}$ is given, the necessary condition for the optimal $\mathbf{Q}$ can be efficiently solved by a Lagrangian method as shown in the following Lemma.

Lemma 1: For fixed $\mathbf{W} = [\mathbf{W}_1\ \mathbf{W}_2]$ with $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{C}^{N_t\times N_d}$, the optimal relay matrix $\mathbf{Q}$ is expressed as $\hat{\mathbf{Q}} = \mathbf{B}\mathbf{L}$ where $\mathbf{B} \in \mathbb{C}^{N_r\times N_t}$ and $\mathbf{L} \in \mathbb{C}^{N_t\times N_r}$ stand for the relay precoder and the receiver, respectively, and are computed as

$$\begin{aligned}\mathbf{B} &= \left(\mathbf{G}^H\mathbf{W}_2^H\mathbf{W}_2\mathbf{G} + \mu\mathbf{I}_{N_r}\right)^{-1}\mathbf{G}^H\mathbf{W}_2^H\left(\mathbf{I}_{N_t} - \mathbf{W}_1\mathbf{T}\right)\\ &\quad\times\left(\mathbf{I}_{N_t} + (\mathbf{H}^H\mathbf{H} + \rho^{-1}\mathbf{I}_{N_t})^{-1}\mathbf{T}^H\mathbf{T}\right)\\ \mathbf{L} &= \left(\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1}\mathbf{H}^H,\end{aligned}$$

where μ is chosen to satisfy the relay constraint in (2).

Proof: See Appendix A. ∎

Note that if we have no direct link, i.e., $\mathbf{T} = \mathbf{0}$, then $\mathbf{B}$ and $\mathbf{L}$ in Lemma 1 are given by simple transmit and receive Wiener filter structures [16] [22]. Again, plugging the result in Lemma 1 into (4), we obtain the equivalent error covariance matrix as a function of $\mathbf{B}$ as illustrated in the following lemma.

Lemma 2: Define a positive semi-definite matrix $\mathbf{\Omega} \in \mathbb{S}_+^{N_t}$ as

$$\mathbf{\Omega} \triangleq \mathbf{L}\mathbf{H}\mathbf{R}_T = \mathbf{L}\left(\mathbf{H}\mathbf{R}_T\mathbf{H}^H + \mathbf{I}_{N_r}\right)\mathbf{L}^H, \tag{5}$$

and its eigenvalue decomposition $\mathbf{\Omega} = \mathbf{U}_\omega\mathbf{\Lambda}_\omega\mathbf{U}_\omega^H$ where $\mathbf{\Lambda}_\omega$ represents a square diagonal matrix with eigenvalues $\lambda_{\omega,k}$ for $k = 1,\ldots,N_t$ arranged in descending order. Then, with the given structure of the relay matrix $\hat{\mathbf{Q}} = \mathbf{B}\mathbf{L}$, the error covariance $\mathbf{R}_e(\hat{\mathbf{Q}})$ in (4) can be decomposed into two individual covariance matrices as

$$\begin{aligned}\mathbf{R}_e(\mathbf{B}) &= \left(\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1}\\ &\quad+ \widetilde{\mathbf{U}}_\omega\left(\widetilde{\mathbf{U}}_\omega^H\mathbf{B}^H\mathbf{G}^H\mathbf{G}\mathbf{B}\widetilde{\mathbf{U}}_\omega + \widetilde{\mathbf{\Lambda}}_\omega^{-1}\right)^{-1}\widetilde{\mathbf{U}}_\omega^H,\end{aligned} \tag{6}$$

where $\widetilde{\mathbf{U}}_\omega \in \mathbb{C}^{N_t\times M}$ denotes a matrix constructed by the first M columns of $\mathbf{U}_\omega$ and $\widetilde{\mathbf{\Lambda}}_\omega = \widetilde{\mathbf{U}}_\omega^H\mathbf{\Omega}\widetilde{\mathbf{U}}_\omega$ indicates the $M \times M$ upper-left submatrix of $\mathbf{\Lambda}_\omega$ where $M \triangleq \min(N_t, N_r)$.

Proof: See Appendix B. ∎

Supposing $M = N_t$ ($N_t \leq N_r$), the second term of (6) can be simplified as $\left(\mathbf{B}^H\mathbf{G}^H\mathbf{G}\mathbf{B} + \mathbf{\Omega}^{-1}\right)^{-1}$. In addition, if the direct link is negligible ($\mathbf{T} = \mathbf{0}$), $\mathbf{\Omega}$ simply represents the covariance matrix of the relay receiver output signal $\mathbf{L}\mathbf{y}_r$ defined as

$$\mathbf{R}_y \triangleq \mathbf{L}(\rho\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})\mathbf{L}^H. \tag{7}$$

This result implies that the derived expression in Lemma 2 generalizes the previous work in [16] to cooperative relaying channels. In fact, in the pure relaying channel, the property of $\mathbf{\Omega} = \mathbf{R}_y$ leads to a simple convex problem which is relatively easy to solve [15] [16]. However, this is obviously not true for cooperative relaying cases. Moreover, noting that the rank of $\mathbf{\Omega}$ equals M, $\mathbf{\Omega}$ becomes rank deficient when $N_t > N_r$. These facts will make the problem more challenging. The detailed behavior of $\mathbf{\Omega}$ will be discussed later in the analysis part in Section V.

We now see from Lemma 2 that the first term in (6) is independent of $\mathbf{Q}$ or $\mathbf{W}$, and thus we only need to optimize the second term of (6) with respect to $\mathbf{B}$. Finally, the original joint optimization problem in (2) can be transformed to a simple problem which finds the optimal relay precoder $\mathbf{B}$ as

$$\min_{\mathbf{B}} \mathrm{Tr}\left(\widetilde{\mathbf{U}}_\omega^H\mathbf{B}^H\mathbf{G}^H\mathbf{G}\mathbf{B}\widetilde{\mathbf{U}}_\omega + \widetilde{\mathbf{\Lambda}}_\omega^{-1}\right)^{-1} \quad s.t.\ \ \mathrm{Tr}\left(\mathbf{B}\mathbf{R}_y\mathbf{B}^H\right) \leq P_R. \tag{8}$$

It is important to note that problem (8) is considerably simplified without any optimality loss compared to (2). This optimality issue will be addressed more in detail in Appendix C.

IV. Closed-form Transceiver Designs

In this section, we solve the problem (8) and derive a new closed-form solution for the relay transceiver $\mathbf{Q}$. First, we define the eigenvalue decomposition $\mathbf{G}^H\mathbf{G} = \mathbf{V}_g\mathbf{\Lambda}_g\mathbf{V}_g^H$ where $\mathbf{\Lambda}_g$ is a square diagonal matrix with eigenvalues $\lambda_{g,k}$ for $k = 1,\ldots,N_r$ in descending order. Then, without loss of generality, we can write $\mathbf{B}$ in (8) in a general form as $\mathbf{B} = \mathbf{V}_g \mathbf{\Phi} \mathbf{U}_\omega^H$ with

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1 & \mathbf{\Phi}_2 \\ \mathbf{\Phi}_3 & \mathbf{\Phi}_4 \end{bmatrix} \in \mathbb{C}^{N_r \times N_t},$$

where the dimension of each submatrix is given by $\mathbf{\Phi}_1 \in \mathbb{C}^{M\times M}$, $\mathbf{\Phi}_2 \in \mathbb{C}^{M\times(N_t-M)}$, $\mathbf{\Phi}_3 \in \mathbb{C}^{(N_r-M)\times M}$, and $\mathbf{\Phi}_4 \in \mathbb{C}^{(N_r-M)\times(N_t-M)}$.

Through some deductions, it is easy to show that the setting $\mathbf{\Phi}_i = \mathbf{0}$ for $i = 2, 3, 4$ has no impact on the objective function[2] in (8) while the power consumption is reduced [13] [15]. Thus, it follows $\hat{\mathbf{B}} = \widetilde{\mathbf{V}}_g \mathbf{\Phi}_1 \widetilde{\mathbf{U}}_\omega^H$, where $\widetilde{\mathbf{V}}_g$ denotes a matrix constructed by the first M columns of $\mathbf{V}_g$. Then, substituting $\hat{\mathbf{B}}$ into (8), the modified problem determines the optimal $\mathbf{\Phi}_1$ as

$$\hat{\mathbf{\Phi}}_1 = \arg\min_{\mathbf{\Phi}_1} \ f_o(\mathbf{\Phi}_1) \quad s.t. \ \ f_p(\mathbf{\Phi}_1) \le P_R, \tag{9}$$

where we have

$$\begin{aligned} f_o(\mathbf{\Phi}_1) &\triangleq \mathrm{Tr}\big(\mathbf{\Phi}_1^H \widetilde{\mathbf{\Lambda}}_g \mathbf{\Phi}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1}\big)^{-1} \\ \text{and} \ \ f_p(\mathbf{\Phi}_1) &\triangleq \mathrm{Tr}(\mathbf{\Phi}_1 \mathbf{R}_\omega \mathbf{\Phi}_1^H). \end{aligned} \tag{10}$$

Here $\widetilde{\mathbf{\Lambda}}_g$ represents the $M \times M$ upper-left submatrix of $\mathbf{\Lambda}_g$ and $\mathbf{R}_\omega \triangleq \widetilde{\mathbf{U}}_\omega^H \mathbf{R}_y \widetilde{\mathbf{U}}_\omega$ is a positive definite matrix.

It is well known that for $\mathbf{A}$ and $\mathbf{B} \in \mathbb{S}_+^M$, $\mathrm{Tr}(\mathbf{AB}) \ge \sum_{i=1}^M \lambda_i(\mathbf{A})\lambda_{M-i+1}(\mathbf{B})$ and $\mathrm{Tr}(\mathbf{A}^{-1}) \ge \sum_{i=1}^M ([\mathbf{A}]_{i,i})^{-1}$ where $\lambda_i(\mathbf{A})$ stands for the i-th largest eigenvalue [23]. From these facts, we can check that the optimum solution for (9) is obtained when the matrices inside the trace in (10), $\mathbf{\Phi}_1^H \widetilde{\mathbf{\Lambda}}_g \mathbf{\Phi}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1}$ and $\mathbf{\Phi}_1 \mathbf{R}_\omega \mathbf{\Phi}_1^H$, are simultaneously diagonalized. However, unlike the case of pure relaying channels where $\mathbf{R}_\omega = \widetilde{\mathbf{\Lambda}}_\omega$ [15] [16], there exists no such a case for all $\mathbf{\Phi}_1$ due to the non-diagonal structure of $\mathbf{R}_\omega$, which makes the problem (9) generally non-convex. To overcome this difficulty and provide a closed-form solution, we impose the following structural constraint on $\mathbf{\Phi}_1$.

Let us define a diagonal matrix $\mathbf{\Phi}_d \in \mathbb{C}^{M\times M}$ with diagonal entries $\phi_1, \ldots, \phi_M$. Then, we can always find a proper $\mathbf{\Phi}_d$ such that $f_o(\mathbf{\Phi}_d) = f_o(\hat{\mathbf{\Phi}}_1)$, since $\mathbf{\Phi}_d$ is the optimum structure for the objective function $f_o(\cdot)$. For the problem (9), however, $\mathbf{\Phi}_d$ is obviously suboptimal, because it may increase the power consumption, i.e., $f_p(\mathbf{\Phi}_d) \ge f_p(\hat{\mathbf{\Phi}}_1)$, with the same MSE. Nevertheless, in this paper, we consider $\mathbf{\Phi}_d$ as a solution, since a diagonal structure allows a simple convex problem and makes the further analysis tractable. Then, by imposing the diagonal structure on the problem (9), we attain the following convex optimization problem[3] as

$$\hat{\mathbf{\Phi}}_d = \arg\min_{\mathbf{\Phi}_d} f_o(\mathbf{\Phi}_d) \quad s.t. \ \ f_p(\mathbf{\Phi}_d) \le P_R. \tag{11}$$

The solution for (11) can be found efficiently by using the Lagrangian multiplier ν.

[2]This is because $\mathbf{\Phi}_2$ and $\mathbf{\Phi}_4$ will not be involved in the objective nor the constraints, and nonzero $\mathbf{\Phi}_3$ always leads to the increased power consumption.

[3]Strictly speaking, the problem (11) is convex with respect to $\mathbf{\Phi}_d \mathbf{\Phi}_d^H$, since the phase of each element in $\mathbf{\Phi}_d$ has no impact on both $f_o(\mathbf{\Phi}_d)$ and $f_p(\mathbf{\Phi}_d)$.

Finally, combining with the relay receiver $\mathbf{L}$ in Lemma 1, a closed-form solution for the relay transceiver $\mathbf{Q}$ is obtained as

$$\mathbf{Q}_{cf} = \widetilde{\mathbf{V}}_g \hat{\mathbf{\Phi}}_d \widetilde{\mathbf{U}}_\omega^H \mathbf{L}.$$

Here, each element of $\hat{\mathbf{\Phi}}_d$ is determined by $|\hat{\phi}_k|^2 = \frac{1}{\lambda_{\omega,k}\lambda_{g,k}} \left(\sqrt{\frac{\lambda_{\omega,k}^2 \lambda_{g,k}}{\nu R_k}} - 1\right)^+$ for $k = 1, \ldots, M$ where $(x)^+$ denotes $\max(x, 0)$, $R_k \triangleq [\mathbf{R}_\omega]_{k,k}$, and ν is chosen to satisfy the power constraint in (11). If $\lambda_{g,k} = 0$, we have $\hat{\phi}_k = 0$ from (11).

It is worthwhile to note that the derived solution becomes globally optimal, i.e., $\hat{\mathbf{\Phi}}_1 = \hat{\mathbf{\Phi}}_d$ when $\mathbf{R}_\omega$ is a diagonal matrix. Accordingly, conventional optimal designs over pure relaying channels where $\mathbf{R}_\omega = \widetilde{\mathbf{\Lambda}}_\omega$ [15] [16] can be regarded as a special case of the proposed scheme. For the same reason, our solution is also optimal for systems with $M = 1$ ($N_t = 1$ or $N_r = 1$), because in this case, we have a positive scalar $\mathbf{R}_\omega$. However, the optimality is not guaranteed in general, since our design strategy ignores the off-diagonal elements of $\mathbf{R}_\omega$ as in (11). Nevertheless, numerical results in Section VI demonstrate that the proposed closed-form solution shows little performance loss compared to the optimal design based on the iterative gradient algorithm [21].

V. Diversity Multiplexing Tradeoff Analysis

In this section, we investigate the analytical performance of the proposed scheme using the DMT analysis. We first propose a DMT upperbound which provides a theoretical limit of MMSE based cooperative relaying systems. Then, we present an achievable DMT of the proposed scheme and find the optimality condition to approach the upperbound. Several definitions and assumptions are given below.

Letting $R(\rho)$ and $P_e(\rho)$ denote the transmit rate and the error probability with the operating SNR ρ, respectively, the multiplexing gain r and the corresponding diversity gain $d(r)$ are defined as [2] [10]

$$\lim_{\rho\to\infty} \frac{R(\rho)}{\log\rho} = r \quad \text{and} \quad \lim_{\rho\to\infty} \frac{\log P_e(\rho)}{\log\rho} = -d(r),$$

and we write $P_e(\rho) \doteq \rho^{-d(r)}$ for notational simplicity. The inequalities $\dot{\le}$ and $\dot{\ge}$ are similarly defined. Note that if the rate $R(\rho)$ is a constant over all SNR range, the multiplexing gain converges to zero. In this paper, the outage probability will be studied, since the outage performance of the mutual information (MI) gives a good approximation of the block error rate [7]–[10]. Also, for simplicity of our analysis, we assume $P_T = P_R = N_t\rho$ and $\sigma_h^2 = \sigma_g^2 = \sigma_t^2 = 1$, but the result can be easily extended to more general cases.

A. Upperbound of DMT

The following theorem provides a DMT performance upperbound.

Theorem 1: For MMSE-based cooperative relaying systems with $N_t \le N_d + N$, an upperbound of DMT is given by

$$d_{\mathrm{ub}}(r) = (N_r + N_d - N_t + 1)\left(1 - \frac{2r}{N_t}\right)^+. \tag{12}$$

Proof: In half duplex relaying systems with the MMSE spatial equalizer and Gaussian input codeword jointly encoded across antennas[4], the MI can be defined as

$$\mathcal{I} = \frac{1}{2}\sum_{k=1}^{N_t}\log\left(1+\gamma_k\right), \tag{13}$$

where $\gamma_k = \rho/[\mathbf{R}_e]_{kk} - 1$. Then, using Jensen's inequality and eliminating the second term of $\mathbf{R}_e$ in (6), the MI is upperbounded by

$$\begin{aligned}\mathcal{I} &\leq \frac{N_t}{2}\log\left(\frac{1}{N_t}\sum_{k=1}^{N_t}\frac{\rho}{[\mathbf{R}_e]_{k,k}}\right)\\ &\leq \frac{N_t}{2}\log\left(\frac{1}{N_t}\sum_{k=1}^{N_t}\frac{1}{(\rho\mathbf{H}_T^H\mathbf{H}_T+\mathbf{I}_{N_t})^{-1}_{k,k}}\right), \end{aligned}\tag{14}$$

where $\mathbf{H}_T \triangleq [\mathbf{H}^T\ \mathbf{T}^T]^T \in \mathbb{C}^{(N_r+N_d)\times N_t}$. Now, we can check that the terms inside the logarithm in (14) exactly coincide with the point-to-point MIMO channel with N_t transmit and N_r+N_d receive antennas, and thus the remaining proof for the outage exponent simply follows the previous result in [25][5]. ■

Note that the multiplexing gain r multiplied by 2 in (12) is attributed to the half-duplex nature of the system, which means that $r \leq N_t/2$. This theorem illustrates a theoretical error performance limit of the MMSE-based cooperative relaying system. In fact, for the majority of cases (e.g., $N_t \leq N_d$ or $N_r < 2N_d$), the upperbound is actually achievable. However, in some specific cases, there still remains a gap between the upperbound and the achievable DMT. A detailed proof will be given in the subsequent subsection.

B. Achievable DMT

Before we address our main result, we introduce several useful lemmas regarding the parameters in problem (11). Let us first consider $\lambda_{\omega,k}$ which is the k-th largest eigenvalue of $\mathbf{\Omega}$ in Lemma 2. Then, supposing a diagonal matrix $\mathbf{\Lambda}_t \in \mathbb{C}^{N_t\times N_t}$ whose diagonal entries consist of the eigenvalues of $\mathbf{T}^H\mathbf{T}$ arranged in *ascending* order, i.e., $\lambda_{t,1} \leq, \lambda_{t,2} \leq, \ldots, \leq \lambda_{t,N_t}$, we can show that $\lambda_{\omega,k}^{-1} \geq \lambda_{t,k}+\rho^{-1}$ for $k=1,\ldots,M$ as proved in the following lemma.

Lemma 3: Letting $\widetilde{\mathbf{\Lambda}}_t$ be the $M\times M$ upper-left submatrix of $\mathbf{\Lambda}_t$, the following inequality holds as

$$\widetilde{\mathbf{\Lambda}}_\omega^{-1} \triangleq (\widetilde{\mathbf{U}}_\omega^H\mathbf{\Omega}\widetilde{\mathbf{U}}_\omega)^{-1} \succeq \widetilde{\mathbf{\Lambda}}_t + \rho^{-1}\mathbf{I}_M,$$

where $\succeq$ (or $\preceq$) represents the generalized inequality defined on the positive semi-definite cone [26].

Proof: After some manipulations, $\mathbf{\Omega}$ in (5) can be modified as $\mathbf{\Omega} = \mathbf{R}_T - \left(\mathbf{H}^H\mathbf{H}+\mathbf{R}_T^{-1}\right)^{-1}$. Since $\mathbf{A}=\mathbf{B}-\mathbf{C}$ implies $\mathbf{A}\preceq\mathbf{B}$ for $\mathbf{A},\ \mathbf{B},\ \mathbf{C}\in\mathbb{S}_+^{N_t}$, it must be true that $\mathbf{\Omega}\preceq\mathbf{R}_T$. Therefore, assuming that the eigenvalues of $\mathbf{R}_T$ are arranged in descending order, we have $\lambda_{\omega,k} \leq (\lambda_{t,k}+\rho^{-1})^{-1}$ for all k, and conversely we obtain $\lambda_{\omega,k}^{-1} \geq \lambda_{t,k}+\rho^{-1}$ for $k=1,\ldots,M$, and the lemma is proved. ■

In addition to Lemma 3, the following lemma will also be useful for simplifying the power-loading matrix $\hat{\mathbf{\Phi}}_d$ proposed in the previous section.

Lemma 4: At high SNR, the k-th diagonal element R_k of $\mathbf{R}_\omega$ in (10) can be upperbounded by $R_k \leq \rho$.

Proof: $\mathbf{R}_\omega$ can be approximated at high SNR as

$$\begin{aligned}&\mathbf{R}_\omega = \rho\widetilde{\mathbf{U}}_\omega^H\mathbf{L}(\mathbf{H}\mathbf{H}^H+\rho^{-1}\mathbf{I}_{N_t})\mathbf{L}^H\widetilde{\mathbf{U}}_\omega\\ &\quad\approx \rho\widetilde{\mathbf{U}}_\omega^H(\mathbf{H}^H\mathbf{H}+\mathbf{R}_T^{-1})^{-1}(\mathbf{H}^H\mathbf{H})^2(\mathbf{H}^H\mathbf{H}+\mathbf{R}_T^{-1})^{-1}\widetilde{\mathbf{U}}_\omega.\end{aligned}$$

Since we have $\mathbf{H}^H\mathbf{H}\preceq\mathbf{H}^H\mathbf{H}+\mathbf{R}_T^{-1}$, it is straightforward that $\mathbf{R}_\omega \preceq \rho\mathbf{I}_M$. Then, by definition of the positive definiteness, we simply obtain the lemma. ■

Now, we characterize an MI lowerbound which describes an achievable DMT of the proposed scheme. Since the function $-\log(\cdot)$ is convex, using the definition in (13) and Jensen's inequality again, we have

$$\begin{aligned}\mathcal{I} &\geq -\frac{N_t}{2}\log\left(\frac{1}{\rho N_t}\mathrm{Tr}\big(\mathbf{R}_e\big)\right)\\ &= -\frac{N_t}{2}\log\left(\frac{1}{\rho N_t}\mathrm{Tr}\big(\hat{\mathbf{\Phi}}_d^H\widetilde{\mathbf{\Lambda}}_g\hat{\mathbf{\Phi}}_d+\widetilde{\mathbf{\Lambda}}_\omega^{-1}\big)^{-1}+\sigma\right),\end{aligned}$$

where $\sigma \triangleq N_t^{-1}\mathrm{Tr}\big(\rho\mathbf{H}_T^H\mathbf{H}_T+\mathbf{I}_{N_t}\big)^{-1}$. Since $\hat{\mathbf{\Phi}}_d$ is optimal under the diagonal structure, the setting $\hat{\mathbf{\Phi}}_d = \sqrt{\eta}\mathbf{I}_M$ clearly yields an MI lowerbound where η can be chosen to be $\eta = P_R/(\rho M)$ from Lemma 4 and the relay constraint in (11).

Then, by the assumption $P_R = N_t\rho$, we have $\eta = N_t/M \geq 1$ and it follows

$$\begin{aligned}\mathcal{I} &\geq -\frac{N_t}{2}\log\left(\frac{1}{\rho N_t}\mathrm{Tr}\Big(\eta\widetilde{\mathbf{\Lambda}}_g+\widetilde{\mathbf{\Lambda}}_\omega^{-1}\Big)^{-1}+\sigma\right)\\ &\geq -\frac{N_t}{2}\log\left(\frac{1}{N_t}\mathrm{Tr}\Big(\rho\widetilde{\mathbf{\Lambda}}_g+\rho\widetilde{\mathbf{\Lambda}}_t+\mathbf{I}_M\Big)^{-1}+\sigma\right),\end{aligned}$$

where the last inequality follows from Lemma 3, because $\mathbf{A}\preceq\mathbf{B}$ implies $\mathrm{Tr}(\mathbf{A}^{-1})\geq\mathrm{Tr}(\mathbf{B}^{-1})$. Note that if the relay power P_R is limited (a fixed constant), the term $\widetilde{\mathbf{\Lambda}}_g$ can be ignored at high SNR ($\rho\to\infty$), since η converges to zero and the direct link component $\widetilde{\mathbf{\Lambda}}_t$ will dominate the performance[6]. The important feature to notice here is that $\widetilde{\mathbf{\Lambda}}_t$ consists of M smallest eigenvalues of $\mathbf{\Lambda}_t$ and is arranged in ascending order in contrast to $\widetilde{\mathbf{\Lambda}}_g$.

Using this bound and setting the target data rate as $R(\rho) = r\log\rho$, we finally obtain the outage probability as

$$\begin{aligned}&P_{\text{out}} \triangleq (\mathcal{I}\leq R(\rho))\\ &\leq P\Big(\mathrm{Tr}\big(\rho\mathbf{H}_T^H\mathbf{H}_T+\mathbf{I}_{N_t}\big)^{-1}\\ &\qquad\qquad+\mathrm{Tr}\big(\rho(\widetilde{\mathbf{\Lambda}}_g+\widetilde{\mathbf{\Lambda}}_t)+\mathbf{I}_M\big)^{-1} \geq N_t\rho^{-\frac{2r}{N_t}}\Big)\\ &= P\left(\sum_{k=1}^{N_t}\frac{1}{1+\rho\lambda_{ht,k}}+\sum_{k=1}^{M}\frac{1}{1+\rho\lambda_{gt,k}} \geq N_t\rho^{-\frac{2r}{N_t}}\right),\end{aligned}\tag{15}$$

where $\lambda_{ht,k}$ designates the k-th largest eigenvalue of $\mathbf{H}_T^H\mathbf{H}_T$ and $\lambda_{gt,k} \triangleq \lambda_{g,k}+\lambda_{t,k}$. The resulting outage exponent is summarized in the following theorem.

[4]This coding strategy is also called *vertical encoding* on which we focus here, but our result can be easily applied to *horizontal encoding* where data streams are separately encoded in each source antenna [24].

[5]Note that here we ruled out the case of a small fixed rate which leads to an additional diversity increment (see [25] for detail). This case is beyond the scope of the paper and currently under investigation for our future work.

[6]This phenomenon is also called the bottleneck effect of the relay [7].

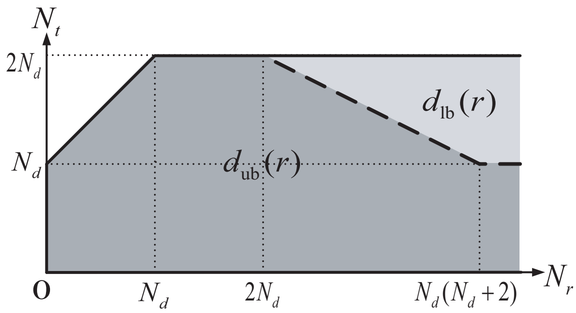

Fig. 2. The achievable DMT region of the proposed scheme where boundary points (dashed line) belong to $d_{\mathrm{ub}}(r)$

Theorem 2: For MMSE based cooperative MIMO relaying channels with $N_t \leq N_d + N$, an achievable DMT of the proposed scheme is characterized as

$$d_{\mathrm{MMSE}}(r) = \begin{cases} d_{\mathrm{ub}}(r), & \text{if } N_t \leq \max(N_d, 2(N_d+1) - \frac{N_r}{N_d}) \\ d_{\mathrm{lb}}(r), & \text{otherwise,} \end{cases}$$

where $d_{\mathrm{lb}}(r) \triangleq (N_d+1)(2N_d - N_t + 1)(1 - \frac{2r}{N_t})^+$ denotes a DMT lowerbound.

Proof: See Appendix D. ∎

The result of Theorem 2 is illustrated in Fig. 2 which shows several interesting observations. First, for $N_t \leq N_d$, the proposed scheme achieves the DMT upperbound for all possible N_r. Also, if $N_r \leq 2N_d$, the optimum tradeoff is always attained for the feasible number of spatial streams N_t. However, in the case of $d_{\mathrm{lb}}(r)$ where both conditions $N_t > N_d$ and $N_r > N_d(2N_d - N_t + 2)$ are satisfied, the proposed scheme does not get the upperbound.

One interesting point in Theorem 2 is that an expression of $d_{\mathrm{lb}}(r)$ is independent of N_r, which means that in the region where $N_t > N_d$, the increase of N_r over $N_d(2N_d - N_t + 2)$ does not introduce any DMT advantage. For example, assuming $N_t = 2N_d$ and $N_r > 2N_d$, the achievable tradeoff equals $d_{\mathrm{lb}}(r) = (N_d + 1)(1 - 2r/N_t)$ regardless of N_r. Therefore, in this case, employing the relay antennas more than $2N_d$ may not be efficient in terms of the DMT. As verified by simulation results in Section VI, our achievable DMT $d_{\mathrm{MMSE}}(r)$ in Theorem 2 accurately predicts the numerical performance of the optimal design [21] as well as the proposed scheme for all cases[7].

On the other hand, as we have $N_t \leq N_d + N$, the achievable multiplexing gain is determined by $r \leq (N_d + N)/2$, which implies that the proposed scheme can support $N_d/2$ higher multiplexing gain compared to the pure relaying systems. The maximum gain $r = N_d$ is achievable for $N_r \geq N_d$, but a small number of relay antennas such as $N_r < N_d$ leads to $r \leq (N_d + N_r)/2$ and may seriously deteriorate the achievable multiplexing gain. Finally, we note that our analytical results also account for the conventional designs with $\mathbf{T} = \mathbf{0}$ [15] [16] or $M = 1$ [19] as special cases. All claims presented in this section will also be confirmed by numerical simulations in the following section.

[7]For this reason, we conjecture that the derived DMT expression $d_{\mathrm{MMSE}}(r)$ actually represents the optimal tradeoff in MMSE-based cooperative relaying systems. A rigorous proof remains open for future work.

VI. Numerical Results

In this section, we present numerical results and comparisons to demonstrate the efficiency of the proposed scheme and support the analysis derived in the previous sections. We compare the following relay matrix designs with the proposed solution.

- *Naive AF*: The most simple scheme where only the power normalizing operation is performed at the relay, i.e., $\mathbf{Q} = \sqrt{P_R/\mathrm{Tr}(\rho \mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})}\mathbf{I}_{N_r}$.
- *Suboptimal*: The optimal solution without considering a direct link proposed in [15], which corresponds to the proposed scheme with setting $\mathbf{T} = \mathbf{0}$.
- *Optimal*: The optimal solution found by an iterative method such as a projected gradient algorithm in [21].

All simulation results have been performed in cooperative MIMO AF relaying channels without CSI at the source. For fairness of comparison, all relaying strategies employ the MMSE receiver at the destination. The notation $N_t \times N_r \times N_d$ is used to denote a system with N_t source, N_r relay and N_d destination antennas. The transmission rate R is measured in bits per channel use (bpcu)[8]. The SNR in each link is defined as $\mathrm{SNR_S} \triangleq \sigma_h^2 P_T$, $\mathrm{SNR_{DL}} \triangleq \sigma_t^2 P_T$, and $\mathrm{SNR_R} \triangleq \sigma_g^2 P_R$ for the source-to-relay, the source-to-destination, and the relay-to-destination link, respectively, and written in dB scale. We also define $\mathrm{SNR_0}$ such that $\mathrm{SNR_0} = \mathrm{SNR_S} = \mathrm{SNR_R}$.

From Figures 3 to 5, we compare the MSE performance of the proposed scheme with conventional methods in cooperative relay channels with various system configurations. For all figures, the naive AF exhibits mostly the worst performance, since it does not utilize the CSI of the system. In addition, we can check that the proposed solution clearly outperforms the suboptimal scheme [15], because the direct link is properly exploited in our solution. Therefore, as the direct link gets stronger, a performance gain obtained from our solution will become significant. All these plots also demonstrate that the proposed closed-form solution achieves the performance almost identical to the optimal iterative design with much reduced complexity.

First, in Fig. 3, we describe the MSE performance as a function of $\mathrm{SNR_{DL}}$ in $4 \times 4 \times 4$ and $4 \times 4 \times 2$ systems. As expected, if the direct link SNR is relatively small, the proposed scheme approaches the suboptimal design which corresponds to a special case of our solution with $\mathbf{T} = \mathbf{0}$. On the contrary, however, we see that as the direct link signal becomes strong, our scheme attains a notable MSE improvement. One interesting observation from this figure is that with a small number of antennas at the destination, the proposed scheme exhibits much better performance than conventional schemes. This is because when $N_t > N_d$, the system performance severely depends on a direct link which provides a supplement to insufficient spatial dimension at the destination. Therefore, considering the direct link is much more significant for the relay matrix design with $N_t > N_d$. As will be shown later, the absence of the direct link CSI at the relay actually incurs a significant diversity loss.

Similar observations can be made in Fig. 4 in which the MSE curves are plotted with $\sigma_h^2 = \sigma_g^2 = 1$ and various input

[8]We transmit $2R$ bits for two time slots.

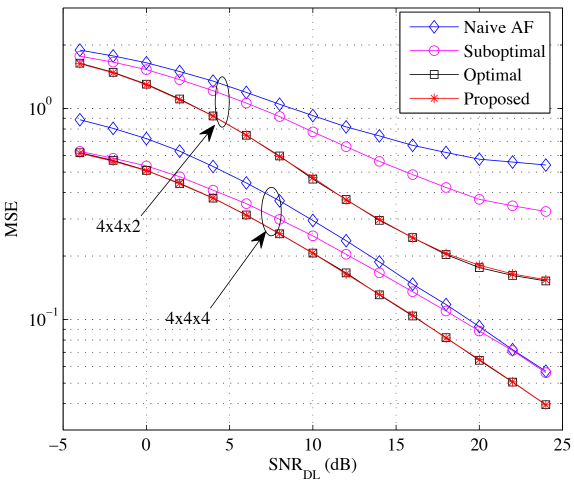

Fig. 3. MSE comparison as a function of $\mathrm{SNR_{DL}}$ with $\mathrm{SNR_0} = 17$ dB.

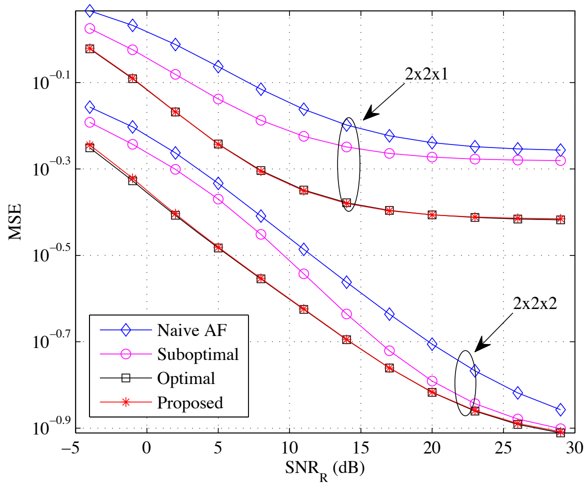

Fig. 5. MSE comparison as a function of $\mathrm{SNR_R}$ with $\mathrm{SNR_S} = 17$ dB and $\mathrm{SNR_{DL}} = 5$ dB.

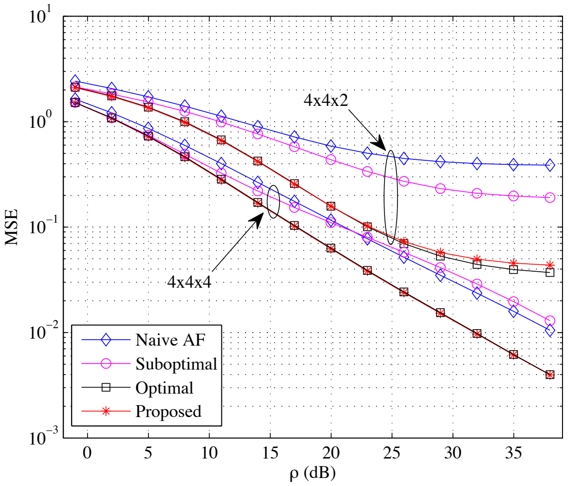

Fig. 4. MSE comparison as a function of input SNR ρ with $\sigma_h^2 = \sigma_g^2 = 1$ where $P_R = 17$ dB and $\mathrm{SNR_{DL}} = \mathrm{SNR_S} - 20$ dB.

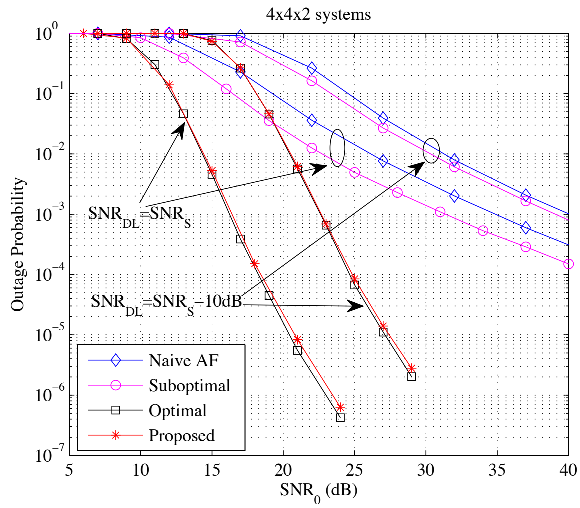

Fig. 6. Outage probability as a function of $\mathrm{SNR_0}$ with $R = 5$ bpcu

SNR ρ. In this case, the source transmit power $P_T = N_t\rho$ and the relay transmit power P_R refer to $\mathrm{SNR_S}$ and $\mathrm{SNR_R}$, respectively. Also, we assume that the direct link suffers from a greater pathloss than the source-to-relay link as $\sigma_t^2 = 0.01$, i.e., $\mathrm{SNR_{DL}} = \mathrm{SNR_S} - 20$ dB. In this situation, it might seem that considering the direct link which is relatively so poor cannot be very useful, but the figure shows that the proposed scheme still achieves a nontrivial MSE advantage in comparison with the conventional schemes even in the moderate input SNR (ρ) range.

In Fig. 5, we describe the MSE performance as a function of $\mathrm{SNR_R}$ in $2\times2\times2$ and $2\times2\times1$ systems. Unlike previous cases, it is shown from curves with $N_d = 2$ that the performance gap between the proposed scheme and the suboptimal scheme reduces as $\mathrm{SNR_R}$ increases. This is due to the fact that the signal coming from the direct link, i.e., $\mathbf{y}_{d_1}$ is relatively negligible compared to the relay signal $\mathbf{y}_{d_2}$. However, it is noteworthy that if $N_t > N_d$, the direct link will significantly affect the performance even if its signal is weak because of the lack of spatial dimension at the destination. This fact explains the reason why the proposed scheme with $N_d = 1$ still outperforms the suboptimal case even in the high $\mathrm{SNR_R}$ range in Fig. 5. From these figures so far, we also confirm that our solution achieves the MSE performance very close to the optimum regardless of SNR in each link.

Next, from Figures 6 to 8, the outage probability achieved by our proposed scheme is compared with that of conventional designs. Fig. 6 depicts the outage performance in $4 \times 4 \times 2$ systems with respect to $\mathrm{SNR_0}$ with a transmit rate $R = 5$ bpcu and various $\mathrm{SNR_{DL}}$. Note that in this case, the multiplexing gain r equals zero. This figure illustrates that our solution obtains a near optimal performance in terms of the outage probability as well. As expected from Theorem 2, it is also shown that both the proposed scheme and the optimal design exhibit the maximum diversity $d_{\mathrm{ub}}(0) = 3$ irrespective of the direct link SNR. The most important thing to notice here is that existing relaying strategies without consideration of the

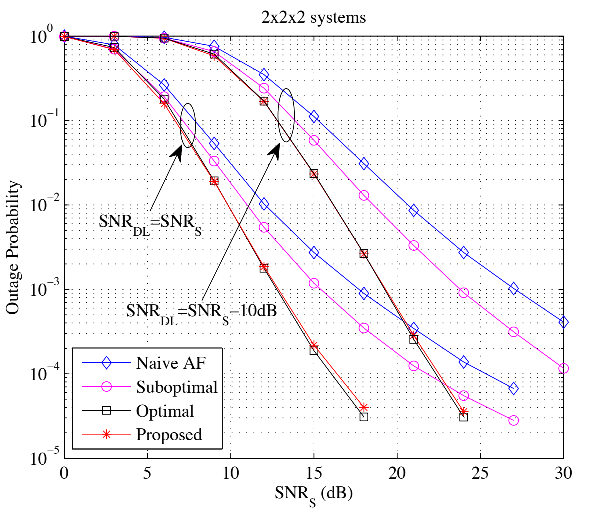

Fig. 7. Outage probability as a function of $\mathrm{SNR_S}$ with $\mathrm{SNR_R} = 10$ dB and $R = 2$ bpcu

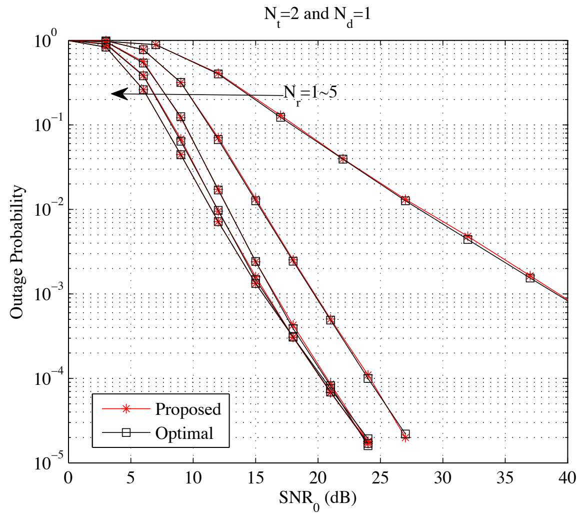

Fig. 9. Outage probability as a function of $\mathrm{SNR_0} = \mathrm{SNR_{DL}}$ with $R = 2$ bpcu

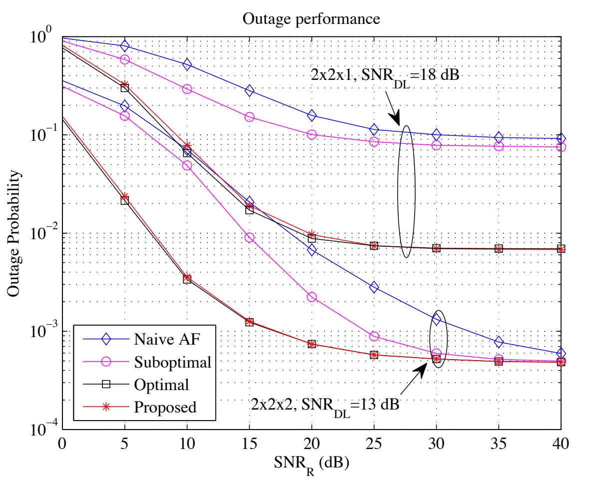

Fig. 8. Outage probability as a function of $\mathrm{SNR_R}$ with $\mathrm{SNR_S} = 23$ dB and $R = 3$ bpcu

direct link experience a severe performance degradation from a diversity loss.

In Fig. 7, the signal power from the relay-to-destination link is limited by $\mathrm{SNR_R} = 10$ dB in $2 \times 2 \times 2$ systems with $R = 2$ bpcu. As described in Section V, the power limitation at the relay incurs a bottleneck effect on the source-relay-destination link. In other words, the signal arrived from the relay i.e., $\mathbf{y}_{d_2}$ is trivial at the destination with high input SNR ρ. Hence, in this example, the diversity order will be eventually determined by the direct link regardless of the relay matrix design. Nevertheless, we observe that the proposed scheme still outperforms existing suboptimal schemes and provides almost the same performance as the optimal design.

Conversely, in Fig. 8, we draw the outage curves with respect to $\mathrm{SNR_R}$ while keeping both $\mathrm{SNR_S}$ and $\mathrm{SNR_{DL}}$ as constants in $2 \times 2 \times 2$ and $2 \times 2 \times 1$ systems. Then, it is seen that diversity orders of all curves converge to zero as $\mathrm{SNR_R}$ goes to high. This observation is easily inferred from the fact that with a so good relay-to-destination channel, the system performance is mostly determined by the broadcast phase in the first time slot. In this case, therefore, the source precoding strategy would be more important, which will be discussed in future work. On the other hand, if $\mathrm{SNR_R}$ is dominant with $N_t \leq N_d$, the direct link signal becomes ignorable at the destination. Therefore, as illustrated from the curves with $N_d = 2$, interestingly all schemes under consideration converge to the same performance. As mentioned before, however, the direct link signal can never be ignored for cases of $N_d = 1$ (i.e., $N_t > N_d$), and thus both the proposed and the optimal relaying strategies which take care of the direct link exhibit an improved outage performance over the others irrespective of $\mathrm{SNR_R}$.

The following two examples illustrate the outage performance with $N_t = 2$ and various numbers of relay antennas to demonstrate the effect of a large number of relay antennas described in Fig. 2. The rate and the direct link SNR are set to be $R = 2$ bpcu and $\mathrm{SNR_{DL}} = \mathrm{SNR_0}$, respectively. Fig. 9 considers the case of $N_t > N_d = 1$. As predicted from our analysis, it is shown that the diversity order is saturated with $d_{\mathrm{lb}}(0) = N_d + 1 = 2$ if $N_r > 2N_d = 2$, while growing according to $d_{\mathrm{ub}}(0) = N_r$ for $N_r \leq 2$. This observation does confirm the accuracy of our achievable DMT expression, $d_{\mathrm{MMSE}}(r)$ in Theorem 2. Another interesting observation here is that if N_r further increases over 3, there will be no performance advantage even in terms of a coding gain. The reason for this is that with a large number of relay antennas, i.e., $N_r > 2N_d + 1$, the minimum channel gain at the second summation in (15) is only associated with the direct link component, i.e., $\min_k \lambda_{gt,k} = \lambda_{t,N_d+1}$. In this case, therefore, implementing the system with $N_r > 3$ could certainly be inefficient. The same comparison is repeated in Fig. 10 with $N_t = N_d = 2$. In contrast to the previous case, we see that the increased number of relay antennas always provides a diversity advantage according to $d_{\mathrm{ub}}(0) = N_r + 1$. These two figures also demonstrate that our analytical results in Theorem 2 precisely estimates the diversity order of the optimal iterative design as well as the proposed scheme.

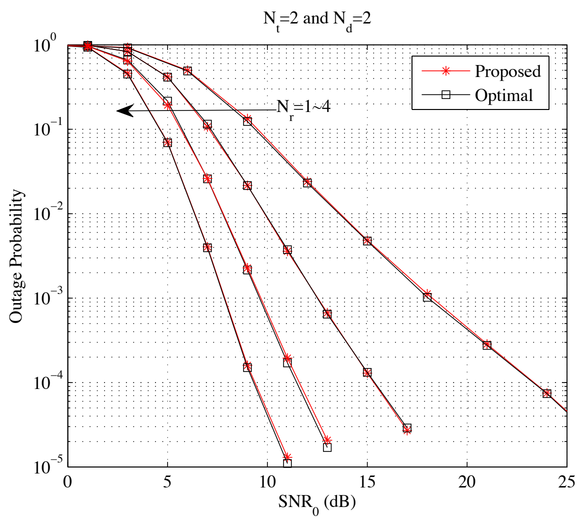

Fig. 10. Outage probability as a function of $\mathrm{SNR}_0 = \mathrm{SNR}_{\mathrm{DL}}$ with $R = 2$ bpcu

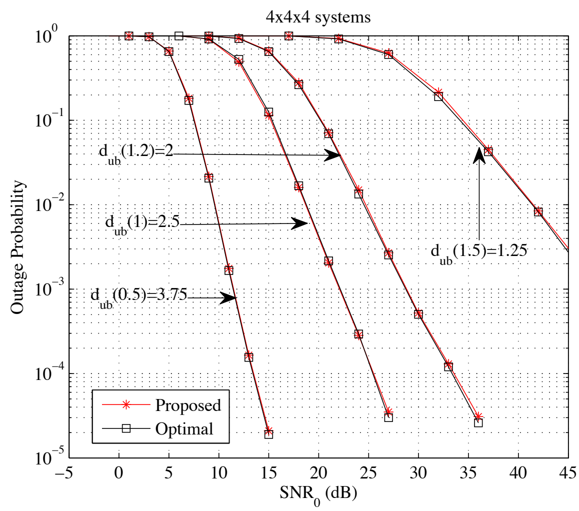

Fig. 11. Outage probability with non-zero multiplexing gain r

Finally, Fig. 11 presents the outage probability of $4 \times 4 \times 4$ systems with a non-zero multiplexing gain r. In other words, for each curve, the transmission rate is set to be an increasing function of SNR as $R(\rho) \doteq r \log \rho$ bpcu. In this case, as predicted from the analysis, our proposed scheme achieves the DMT upperbound which is given by $d_{\mathrm{ub}}(r) = 5(1 - r/2)$. For all simulations in this figure, we confirm that the diversity orders for various multiplexing gain r are well matched with our analytical DMT results.

VII. Conclusion

In this paper, we have considered a closed-form design of the relaying matrix in MMSE-based cooperative MIMO relaying systems. As the problem is complicated and generally non-convex, we have employed the decomposable property of the error covariance matrix and a relaxation technique imposing a structural constraint on the problem to obtain an insightful closed-form solution. Our method also allows us to derive the tradeoff between the diversity and the multiplexing gains of cooperative relaying systems based on the MMSE criterion. From our analysis, several interesting observations have been made. Through numerical simulations, we have verified our analysis and it is shown that the proposed scheme outperforms existing suboptimal schemes and achieves near optimal performance with much reduced complexity compared to the iterative optimal scheme.

Appendix A
Proof of Lemma 1

For a complex matrix $\mathbf{A}$, we first define $\Re\{\mathbf{A}\}$ and $\mathrm{vec}(\mathbf{A})$ as the real part of $\mathbf{A}$ and the stacked columns of $\mathbf{A}$, respectively. Then, using the Lagrangian multiplier μ, we can set up the cost function $\mathcal{C}$ as

$$\begin{aligned}\mathcal{C} &= \mathrm{Tr}\left(\mathbf{R}_e(\mathbf{W}, \mathbf{Q})\right) + \mu \mathrm{Tr}\left(\mathbf{Q}^H(\sigma_x^2 \mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})\mathbf{Q}\right) \\ &= \mathrm{Tr}\big(2\Re[\sigma_x^2 \mathbf{W}_2 \mathbf{G}\mathbf{Q}\mathbf{H}(\mathbf{T}^H\mathbf{W}_1^H + \mathbf{I}_{N_t})] \\ &\quad + \mathbf{W}_2\mathbf{G}\mathbf{Q}(\sigma_x^2\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})\mathbf{Q}^H\mathbf{G}^H\mathbf{W}_2^H \\ &\quad + \mu\mathbf{Q}(\sigma_x^2\mathbf{H}\mathbf{H}^H + \mathbf{I}_{N_r})\mathbf{Q}^H + \mathbf{C}\big),\end{aligned}$$

where the matrix $\mathbf{C}$ indicates the constant terms independent of $\mathbf{Q}$.

Now, we find $\hat{\mathbf{Q}}$ by setting the gradient with respect to $\mathbf{Q}$ to zero as

$$\begin{aligned}\frac{\partial \mathcal{C}}{\partial \mathbf{Q}^*} &= \sigma_x^2 \mathbf{G}^H\mathbf{W}_2^H\mathbf{W}_1\mathbf{T}\mathbf{H}^H + \sigma_x^2\mathbf{G}^H\mathbf{W}_2^H\mathbf{W}_2\mathbf{G}\mathbf{Q}\mathbf{H}\mathbf{H}^H \\ &\quad + \mathbf{G}^H\mathbf{W}_2^H\mathbf{W}_2\mathbf{G}\mathbf{Q} - \sigma_x^2\mathbf{G}_H\mathbf{W}_2^H\mathbf{H}^H \\ &\quad + \mu(\mathbf{Q}\mathbf{H}\mathbf{H}^H + \mathbf{Q}) = 0. \end{aligned} \tag{16}$$

This result can be verified using some rules such as $d\mathrm{tr}\,(\mathbf{Y}) = \mathrm{tr}\,(d\mathbf{Y})$, $\mathrm{vec}(d\mathbf{X}) = d\mathrm{vec}(\mathbf{X})$, $\mathrm{tr}(\mathbf{X}^T\mathbf{Y}) = \mathrm{vec}(\mathbf{X})^T\mathrm{vec}(\mathbf{Y})$ [27]. Finally, solving the equation (16), we obtain the optimal $\mathbf{Q}$ as

$$\begin{aligned}\hat{\mathbf{Q}} &= \left(\mathbf{G}^H\mathbf{W}_2^H\mathbf{W}_2\mathbf{G} + \mu\mathbf{I}_{N_r}\right)^{-1}\mathbf{G}^H\mathbf{W}_2^H(\mathbf{I} - \mathbf{W}_1\mathbf{T}) \\ &\quad \times (\mathbf{H}^H\mathbf{H} + \sigma_x^{-2}\mathbf{I}_{N_t})^{-1}\mathbf{H}^H \\ &= \left(\mathbf{G}^H\mathbf{W}_2^H\mathbf{W}_2\mathbf{G} + \mu\mathbf{I}_{N_r}\right)^{-1}\mathbf{G}^H\mathbf{W}_2^H(\mathbf{I} - \mathbf{W}_1\mathbf{T}) \\ &\quad \times \left(\mathbf{I} + (\mathbf{H}^H\mathbf{H} + \sigma_x^{-2}\mathbf{I}_{N_t})^{-1}\mathbf{T}^H\mathbf{T}\right) \\ &\quad \times \left(\mathbf{H}^H\mathbf{H} + \mathbf{T}^H\mathbf{T} + \sigma_x^{-2}\mathbf{I}_{N_t}\right)^{-1}\mathbf{H}^H\end{aligned}$$

and the proof is concluded.

Appendix B
Proof of Lemma 2

First, substituting the result in Lemma 1, i.e., $\hat{\mathbf{Q}} = \mathbf{B}\mathbf{L}$ into $\mathbf{R}_e(\mathbf{Q})$ in (4), we have

$$\mathbf{R}_e = \left(\mathbf{H}^H\mathbf{L}^H\mathbf{B}^H\mathbf{G}^H\mathbf{R}_n^{-1}\mathbf{G}\mathbf{B}\mathbf{L}\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1}.$$

Then, using the definition of $\mathbf{\Omega} = \mathbf{L}\mathbf{H}\mathbf{R}_T$ in Lemma 2 and invoking the matrix inversion lemma, it follows

$$\begin{aligned}\mathbf{R}_e &= \mathbf{R}_T - \mathbf{\Omega}^H\mathbf{B}^H\mathbf{G}^H \\ &\quad \times \left(\mathbf{G}\mathbf{B}\mathbf{L}\mathbf{H}\mathbf{R}_T\mathbf{H}^H\mathbf{L}^H\mathbf{B}^H\mathbf{G}^H + \mathbf{R}_n\right)^{-1}\mathbf{G}\mathbf{B}\mathbf{\Omega} \\ &= \mathbf{R}_T - \mathbf{\Omega}^H\mathbf{B}^H\mathbf{G}^H \\ &\quad \times \left(\mathbf{G}\mathbf{B}\mathbf{\Omega}\mathbf{B}^H\mathbf{G}^H + \mathbf{I}_{N_d}\right)^{-1}\mathbf{G}\mathbf{B}\mathbf{\Omega}.\end{aligned} \tag{17}$$

Now, we write the relay precoder $\mathbf{B}$ in a more general form as $\mathbf{B} = \breve{\mathbf{B}}\mathbf{U}_\omega^H$ where $\breve{\mathbf{B}} = [\mathbf{B}_1 \;\; \mathbf{B}_2]$ with $\mathbf{B}_1 \in \mathbb{C}^{N_r \times M}$

and $\mathbf{B}_2 \in \mathbb{C}^{N_r \times (N_t - M)}$. Since $\mathbf{\Omega}$ and $\mathbf{R}_y$ in (7) are rank M matrices and share the same null space, the setting $\mathbf{B}_2 = \mathbf{0}$ has no impact on both the MSE in (17) and the relay power consumption in (2), i.e., $\mathrm{Tr}(\mathbf{B}\mathbf{R}_y\mathbf{B}^H)$. Therefore, without loss of generality, $\mathbf{R}_e$ can be rephrased by

$$\mathbf{R}_e = \mathbf{R}_T - \mathbf{\Gamma}_\omega^H \mathbf{B}_1^H \mathbf{G}^H \times \left(\mathbf{G}\mathbf{B}_1\widetilde{\mathbf{\Lambda}}_\omega \mathbf{B}_1^H \mathbf{G}^H + \mathbf{I}_{N_d}\right)^{-1} \mathbf{G}\mathbf{B}_1 \mathbf{\Gamma}_\omega \quad (18)$$

where $\mathbf{\Gamma}_\omega \triangleq \widetilde{\mathbf{U}}_\omega^H \mathbf{\Omega}$ and $\widetilde{\mathbf{U}}_\omega$ is defined in Lemma 2.

Again, applying the matrix inversion lemma to equation (18) and using some mathematical manipulations, we have

$$\begin{aligned}\mathbf{R}_e &= \mathbf{R}_T - \mathbf{\Gamma}_\omega^H \mathbf{B}_1^H \mathbf{G}^H \\ &\quad \times \left(\mathbf{I}_{N_d} - \mathbf{G}\mathbf{B}_1(\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1}\mathbf{B}_1^H\mathbf{G}^H\right) \\ &\quad \times \mathbf{G}\mathbf{B}_1\mathbf{\Gamma}_\omega \\ &= \mathbf{R}_T - \mathbf{\Gamma}_\omega^H \mathbf{B}_1^H \mathbf{G}^H \mathbf{G}\mathbf{B}_1 \mathbf{\Gamma}_\omega \\ &\quad + \mathbf{\Gamma}_\omega^H \mathbf{B}_1^H \mathbf{G}^H \mathbf{G}\mathbf{B}_1 (\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \\ &\quad \times \left(\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1} - \widetilde{\mathbf{\Lambda}}_\omega^{-1}\right)\mathbf{\Gamma}_\omega \\ &= \mathbf{R}_T - \mathbf{\Gamma}_\omega^H \mathbf{B}_1^H \mathbf{G}^H \mathbf{G}\mathbf{B}_1 \\ &\quad \times (\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \widetilde{\mathbf{\Lambda}}_\omega^{-1} \mathbf{\Gamma}_\omega \\ &= \mathbf{R}_T - \mathbf{\Gamma}_\omega^H (\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1} - \widetilde{\mathbf{\Lambda}}_\omega^{-1}) \\ &\quad \times (\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \widetilde{\mathbf{\Lambda}}_\omega^{-1} \mathbf{\Gamma}_\omega \\ &= \mathbf{R}_T - \mathbf{\Gamma}_\omega^H \widetilde{\mathbf{\Lambda}}_\omega^{-1} \mathbf{\Gamma}_\omega \\ &\quad + \mathbf{\Gamma}_\omega^H \widetilde{\mathbf{\Lambda}}_\omega^{-1} (\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \widetilde{\mathbf{\Lambda}}_\omega^{-1} \mathbf{\Gamma}_\omega \\ &= \mathbf{R}_T - \mathbf{\Omega} + \widetilde{\mathbf{U}}_\omega (\mathbf{B}_1^H\mathbf{G}^H\mathbf{G}\mathbf{B}_1 + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \widetilde{\mathbf{U}}_\omega^H,\end{aligned}$$

where the last equality follows from the facts that $\mathbf{\Gamma}_\omega^H \widetilde{\mathbf{\Lambda}}_\omega^{-1} \mathbf{\Gamma}_\omega = \mathbf{\Omega}$ and $\widetilde{\mathbf{\Lambda}}_\omega^{-1} \mathbf{\Gamma}_\omega = \widetilde{\mathbf{U}}_\omega^H$. Then, we finally obtain

$$\begin{aligned}\mathbf{R}_e &= \mathbf{R}_T - \mathbf{R}_T \mathbf{H}^H \mathbf{L}^H \\ &\quad + \widetilde{\mathbf{U}}_\omega (\widetilde{\mathbf{U}}_\omega^H \mathbf{B}^H \mathbf{G}^H \mathbf{G} \mathbf{B} \widetilde{\mathbf{U}}_\omega + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \widetilde{\mathbf{U}}_\omega^H \\ &= \left(\mathbf{H}^H\mathbf{H} + \mathbf{R}_T^{-1}\right)^{-1} \\ &\quad + \widetilde{\mathbf{U}}_\omega (\widetilde{\mathbf{U}}_\omega^H \mathbf{B}^H \mathbf{G}^H \mathbf{G} \mathbf{B} \widetilde{\mathbf{U}}_\omega + \widetilde{\mathbf{\Lambda}}_\omega^{-1})^{-1} \widetilde{\mathbf{U}}_\omega^H\end{aligned}$$

and the proof is completed.

APPENDIX C
OPTIMALITY OF PROBLEM (8)

Let us consider an arbitrary objective function $f(\mathbf{Q}, \mathbf{W})$ and define $\hat{\mathbf{Q}}(\mathbf{W}) \triangleq \arg\min_{\mathbf{Q}} f(\mathbf{Q}, \mathbf{W})$ and $\hat{\mathbf{W}}(\mathbf{Q}) \triangleq \arg\min_{\mathbf{W}} f(\mathbf{Q}, \mathbf{W})$. Then, by the optimization theory [26], we have

$$\min_{\mathbf{Q},\mathbf{W}} f(\mathbf{Q}, \mathbf{W}) = \min_{\mathbf{Q}} f\big(\mathbf{Q}, \hat{\mathbf{W}}(\mathbf{Q})\big) = \min_{\mathbf{W}} f\big(\hat{\mathbf{Q}}(\mathbf{W}), \mathbf{W}\big), \quad (19)$$

which means that we can always minimize a function by first minimizing over some of the variables while fixing others, and then minimizing over the remaining ones. From (19), one can also easily show that $\min_{\mathbf{W}} f\big(\hat{\mathbf{Q}}(\mathbf{W}), \hat{\mathbf{W}}\big(\hat{\mathbf{Q}}(\mathbf{W})\big)\big) = \min_{\mathbf{W}} f\big(\hat{\mathbf{Q}}(\mathbf{W}), \mathbf{W}\big)$, which implies that plugging $\hat{\mathbf{Q}}(\mathbf{W})$ back into $f\big(\mathbf{Q}, \hat{\mathbf{W}}(\mathbf{Q})\big)$ and making the problem a function of $\mathbf{W}$ do not lose the optimality.

Our problem formulation in Section III is performed based on this simple and general optimization principle. Here, $f(\mathbf{Q}, \hat{\mathbf{W}}(\mathbf{Q}))$ corresponds to $\mathrm{tr}(\mathbf{R}_e(\mathbf{Q}))$ in equation (4) and the main purpose of Lemma 1 is to identify $\hat{\mathbf{Q}}(\mathbf{W})$. Accordingly, substituting $\hat{\mathbf{Q}}(\mathbf{W})$ into $\mathbf{R}_e(\mathbf{Q})$, i.e., the result in Lemma 2 has no optimality loss and yields the following equalities as

$$\min_{\mathbf{Q}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{Q})\big) = \min_{\mathbf{W}} \mathrm{tr}\big(\mathbf{R}_e(\hat{\mathbf{Q}}(\mathbf{W}))\big) = \min_{\mathbf{W}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{B})\big). \quad (20)$$

Here, $\mathbf{B}$ and $\mathbf{R}_e(\mathbf{B})$ are defined in Lemma 1 and 2, respectively. Note that for the sake of simplicity, the power constraints of the problems in (20) were omitted.

Now, the remaining work is to show that two problems $\alpha = \min_{\mathbf{W}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{B})\big)$ and $\beta = \min_{\mathbf{B}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{B})\big)$ achieves the same optimal point. First, we note that $\mathbf{B}$ is a function of $\mathbf{W}$, which means that if $\mathbf{W}$ solves the former problem, then a proper $\mathbf{B}$ can always be found to achieve α in the latter problem. Thus, it follows $\alpha \geq \beta$. Similarly, supposing $\gamma = \min_{\mathbf{Q}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{Q})\big)$, it is also clear that $\beta \geq \gamma$, since $\mathbf{Q}$ is a function of $\mathbf{B}$. However, the equality $\alpha = \gamma$ follows from (20), and consequently we obtain $\beta = \gamma$, i.e., $\min_{\mathbf{Q}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{Q})\big) = \min_{\mathbf{B}} \mathrm{tr}\big(\mathbf{R}_e(\mathbf{B})\big)$. This implies that the resulting simplified problem in (8) is indeed equivalent to the original joint problem in (2) and the optimality is maintained.

APPENDIX D
PROOF OF THEOREM 2

First, we introduce the following two lemmas which are useful for the proof.

Lemma 5 ([28]): : For K positive random variables $\{X_i\}_{i=1,\ldots,K}$[9], define $W \triangleq \min(X_1, X_2, \ldots, X_K)$. Then, for a small argument w (i.e., $w \to 0^+$), the CDF of W is given by $F_W(w) = F_{X_1}(w) + F_{X_2}(w) + \cdots + F_{X_K}(w)$ where $F_{X_i}(\cdot)$ indicates the CDF of X_i.

Lemma 6: Let us define two independent and polynomially distributed random variables $Y \geq 0$ and $Z \geq 0$, i.e., $F_Y(y) = c_1 y^\alpha$ and $F_Z(z) = c_2 z^\beta$ where $\alpha \geq 1$ and $\beta \geq 1$, and c_1 and c_2 are constants. Then, the CDF of $S = Y + Z$ equals $F_S(s) = \kappa s^{\alpha+\beta}$ where $\kappa = \frac{c_1 c_2 \alpha! \beta!}{(\alpha+\beta)!}$.

Proof: The CDF of S can be written in a convolution form as $F_S(s) = \int_0^s \alpha c_1 c_2 y^{\alpha-1}(s-y)^\beta dy$. By solving the integral, we simply obtain the result. Details are trivial, and thus skipped. ■

Now, we prove Theorem 2. The outage probability in (15) depends on the worst case channel gain and thus, can be asymptotically upperbounded as

$$\begin{aligned}&P\left(\mathcal{I} \leq R(\rho)\right) \\ &\quad \dot{\leq} P\left(\frac{1}{\rho\lambda_{ht,N_t}} + \frac{1}{\min\limits_{k=1,\ldots,M}\{\rho\lambda_{gt,k}\}} \geq N_t \rho^{-\frac{2r}{N_t}}\right) \\ &\quad \leq P\left(\Delta \leq \frac{2}{N_t}\rho^{-\left(1-\frac{2r}{N_t}\right)}\right), \quad (21)\end{aligned}$$

where $\Delta \triangleq \min(\lambda_{ht,N_t}, \lambda_{gt,\min})$ and $\lambda_{gt,\min} \triangleq \min_{k=1,\ldots,M}\{\lambda_{gt,k}\}$. The last inequality follows from

[9]Here, the random variables $X_1, \ldots, X_K$ do not need to be independent.

the harmonic mean bound $A^{-1} + B^{-1} \leq \frac{2}{\min(A,B)}$. We see that for the case of $2r/N_t \geq 1$, the outage exponent converges to zero. Hence, supposing $2r/N_t < 1$, we find the near zero behavior of a distribution of Δ in the following.

For a small argument δ, we can recognize from Lemma 5 that the CDF of Δ is given by

$$F_\Delta(\delta) = F_{\lambda_{ht,N_t}}(\delta) + \sum_{k=1}^{M} F_{\lambda_{gt,k}}(\delta),$$

where $F_{\lambda_{ht,N_t}}(\cdot)$, and $F_{\lambda_{gt,k}}(\cdot)$ represent the CDFs of λ_{ht,N_t} and $\lambda_{gt,k}$, respectively. It is also well known [29] [30] that for an $m \times m$ complex Wishart matrix $\mathbf{S}^H\mathbf{S}$ with a Gaussian matrix $\mathbf{S} \in \mathbb{C}^{n\times m}$, its k-th largest (or smallest) eigenvalue λ_k is polynomially distributed near zero as $F_{\lambda_k}(\lambda_k) \propto \lambda_k^{(m-k+1)^+(n-k+1)^+}$ (or $\propto \lambda_k^{k(n-m+k)^+}$).

Using these facts and employing Lemma 6, we obtain $F_\Delta(\delta)$ as[10]

$$\begin{aligned} F_\Delta(\delta) &\propto \delta^{(N_r+N_d-N_t+1)} \\ &\quad + \sum_{k=1}^{M} \delta^{(N_d-k+1)^+(N_r-k+1)^+ \,+\, k(N_d-N_t+k)^+} \\ &\simeq \delta^{\min(D_h,D_g)}. \end{aligned}$$

Here, we define

$$\begin{aligned} D_h &\triangleq N_r + N_d - N_t + 1 \\ D_g &\triangleq \min_{k=1,\ldots,M} \big((N_d - k + 1)^+(N_r - k + 1) \\ &\qquad + k(N_d - N_t + k)^+\big). \end{aligned} \tag{22}$$

Then, the outage probability upperbound is readily acquired by

$$P\left(\mathcal{I} \leq R(\rho)\right) \dot{\leq} c\rho^{-\min(D_h,D_g)\left(1-\frac{2r}{N_t}\right)^+},$$

where c is a constant and the resulting outage exponent is $d_{\text{MMSE}}(r) = \min(D_h, D_g)(1 - 2r/N_t)^+$.

Now, we find $d_{\text{MMSE}}(r)$ in a more explicit form. First, we evaluate the diversity order for the case of $N_d \geq M$. In this case, D_g in (22) can be modified as

$$\begin{aligned} D_g &= \min_{k=1,\ldots,M} \big(\overline{D}_h + (N_r - k + 1)(N_d - k) \\ &\qquad + (k-1)(N_d - N_t + k)^+\big), \end{aligned}$$

where $\overline{D}_h \triangleq N_r - k + 1 + (N_d - N_t + k)^+$. Since we have $\overline{D}_h \geq D_h$, it must be true that $\min(D_h, D_g) = D_h$ for $N_d \geq M$. Next, we examine the case of $N_d < M$. For convenience, let us set $N_t = 2N_d - \beta$ for $\beta = 0, 1, \ldots, N_d - 1$. Then, through some deduction, we can show that the minimum D_g occurs only at $k = N_d - m + 1$ for $m = 0, 1, \ldots, \beta + 1$. Therefore, D_h and D_g in (22) can be rewritten as $D_h = N_r - N_d + \beta + 1$ and $D_g = \min_{m=0,\ldots,\beta+1} \Gamma(m)$ where $\Gamma(m) \triangleq m(N_r - N_d + m) + (\beta - m + 1)(N_d - m + 1)$.

Then, it follows

$$\begin{aligned} \Gamma(m) &= D_h + (\beta - m + 1)(N_d - m) \\ &\qquad + (m-1)(N_r - N_d + m) \\ &\geq D_h, \ \forall m \neq 0. \end{aligned}$$

[10] Recall that $\lambda_{gt,k} = \lambda_{g,k} + \lambda_{t,k}$ where $\lambda_{g,k}$ and $\lambda_{t,k}$ are the k-th largest and the k-th smallest eigenvalue of $\mathbf{G}^H\mathbf{G}$ and $\mathbf{T}^H\mathbf{T}$, respectively.

As a result, only when $\Gamma(0) < D_h$, i.e., $N_t > 2(N_d + 1) - N_r/N_d$, the diversity gain is determined by $D_g = \Gamma(0) = (2N_d - N_t + 1)(N_d + 1)$ and otherwise, we have $\min(D_h, D_g) = D_h$. In summary, if both conditions $N_d < M$ and $N_t > 2(N_d + 1) - N_r/N_d$ are satisfied, we obtain $\min(D_h, D_g) = D_g$ and if not, we have $\min(D_h, D_g) = D_h$ and the proof is concluded.

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