Achievable Degree of Freedom on K-User Y Channels with Heterogeneous Messages

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***Abstract*—In this letter, we introduce a new system model for a multi-way communication where multiple users exchange information with each other by sharing an intermediate relay with multiple antennas. Particularly, we focus on the system where each user intends to send multiple unicast messages for dedicated users and a single multicast message for a group of users. We refer to this system as a K-user MIMO Y channel with heterogenous messages. For this channel, we propose a new cooperative beamforming method and study an achievable degree of freedom (DOF) using this scheme.**

***Index Terms*—Degree of freedom, Y channels.**

I. Introduction

DUE to its broad application including cellular networks, ad-hoc networks, and device-to-device communications, there has been a lot of interest in multi-way relay communications [1]. Depending on the applications, two different types of messages can be considered in multi-way relay channels: unicast and multicast message. Unicast message is intended for dedicated users, which implies that all the other users see this unicast message as interference signal. On the other hand, multicast message is used for a group of users, and it is considered as a desired message for the group.

Recently, generalized network models of the two-way relay channel have been studied for two types of messages settings in [1]–[5]. By considering multiple multicast messages, an achievable rate was examined for a multi-way relay channel in [1] where multiple clusters consisting of multiple nodes exchange multicast messages among themselves. For the multiple unicast messages exchange scenario, the multiple-input multiple-output (MIMO) Y channel was introduced in [2] where three users exchange independent unicast messages with each other via an intermediate relay. Also, an asymptotical capacity behavior was studied from a degree of freedom (DOF) perspective. This result was extended into the case of a general number of users as a K-user Y channel in [3]. Moreover, the optimization of the beamforming was studied in [4] and an asymmetric setup was considered in [5].

In this paper, we consider a heterogeneous message exchange scenario in multi-way communication systems where K users having multiple antennas exchange both unicast and multicast messages with each other. We refer to this system as a K*-user MIMO Y channel with heterogenous messages*.

II. System Model

In this section, we describe the system model for the K-user MIMO Y channel with heterogenous messages where K users with M antennas exchange information with each other by using an intermediate relay node with N antennas. Specifically, each user tries to send both unicast (private) and multicast (common) messages to the other users in the multiple access channel (MAC) phase and wishes to decode both of them in the broadcast channel (BC) phase. We denote $W_p^{[j,i]}$ as the private message sent by user i to user j and $W_c^{[i]}$ as the common message transmitted by user i to all other users.

During the MAC phase, all messages $W_p^{[j,i]}$ and $W_c^{[i]}$ are sent by user i using the symbols $s_p^{[j,i]}$ and $s_c^{[i]}$. When the messages are transmitted, user i employs the precoding vectors $\mathbf{v}_p^{[j,i]}$ and $\mathbf{v}_c^{[i]}$ for $s_p^{[j,i]}$ and $s_c^{[i]}$, respectively. Thus, the transmit signal for user i is expressed as $\mathbf{x}^{[i]} = \sum_{j\neq i} \mathbf{v}_p^{[j,i]} s_p^{[j,i]} + \mathbf{v}_c^{[i]} s_c^{[i]}$, which satisfies average power constraint $\mathbb{E}\{\mathrm{tr}[\mathbf{x}^{[i]}\mathbf{x}^{[i]H}]\} \leq P$. Then, the received signal at the relay is given by

$$\mathbf{y}^{[R]} = \sum_{i=1}^{K} \mathbf{H}^{[R,i]}\mathbf{x}^{[i]} + \mathbf{n}^{[R]}, \tag{1}$$

where $\mathbf{H}^{[R,i]}$ represents the $N \times M$ channel matrix from user i to the relay, and $\mathbf{n}^{[R]}$ indicates the additive white Gaussian noise (AWGN) vector with zero mean and unit variance. The relay subsequently generates the transmit signal $\mathbf{x}^{[R]} = \mathbf{F}\mathbf{y}^{[R]}$ where $\mathbf{F} = \gamma\bar{\mathbf{F}}$ is designed by using the beamforming matrix $\bar{\mathbf{F}}$ and applying the power normalizing factor $\gamma = \sqrt{P/\mathbb{E}\{\mathrm{tr}[\bar{\mathbf{F}}\mathbf{y}^{[R]}\mathbf{y}^{[R]H}\bar{\mathbf{F}}^H]\}}$ to satisfy the relay power constraint P.

In the BC phase, the relay broadcasts $\mathbf{x}^{[R]}$ to all users. Then, user j receives the signal as

$$\mathbf{y}^{[j]} = \mathbf{H}^{[j,R]}\mathbf{x}^{[R]} + \mathbf{n}^{[j]}$$

where $\mathbf{H}^{[j,R]}$ and $\mathbf{n}^{[j]}$ denote the $M \times N$ channel matrix from the relay to user j and the AWGN vector at user j, respectively. Throughout this paper, all channel elements are generated from an independent and identically distributed complex Gaussian distribution with zero mean and unit variance. In addition, it is assumed that all users and the relay operate in the full-duplex mode[1] where all nodes can receive and transmit simultaneously.

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[1]Although in our DOF computation, we assume that the relay operates in the full-deplex mode, it can be easily shown that the DOF is reduced by the factor of two if the half-duplex mode is employed.

From the received signal $\mathbf{y}^{[j]}$, user j adopts the receive combining vector $\mathbf{w}_p^{[j,i]}$ and $\mathbf{w}_c^{[j,i]}$ to decode the user i's message $W_p^{[j,i]}$ and $W_c^{[i]}$, respectively, as $\hat{y}_m^{[j,i]} = \mathbf{w}_m^{[j,i]^H} \mathbf{y}^{[j]}$ for $m \in \{p, c\}$. The signal to interference-plus-noise ratio (SINR) for the private message $W_p^{[j,i]}$ is

$$\mathrm{SINR}_p^{[j,i]} = \frac{\left| \mathbf{w}_p^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F} \mathbf{H}^{[R,i]} \mathbf{v}_p^{[j,i]} \right|^2}{I_p^{[j,i]} + N_p^{[j,i]}} \tag{2}$$

where $I_p^{[j,i]} = \sum_{k \neq j} \sum_{(l,k) \neq (j,i)} \left| \mathbf{w}_p^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F} \mathbf{H}^{[R,k]} \mathbf{v}_p^{[l,k]} \right|^2 + \sum_{k \neq j} \left| \mathbf{w}_p^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F} \mathbf{H}^{[R,k]} \mathbf{v}_c^{[k]} \right|^2$ denotes the interference term and $N_m^{[j,i]} = \|\mathbf{w}_m^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F}\|^2 + \|\mathbf{w}_m^{[j,i]^H}\|^2$ indicates the noise power term.

When the interference is treated as noise, the achievable rate for $W_p^{[j,i]}$ becomes $R_p^{[j,i]} = \log_2 \left(1 + \mathrm{SINR}_p^{[j,i]}\right)$. The achievable rate for the common message $W_c^{[i]}$ is obtained by $R_c^{[i]} = \min_{\forall j \neq i} \left\{ R_c^{[j,i]} \right\}$, where $R_c^{[j,i]}$ is the achievable rate of $W_c^{[i]}$ from user i to user j, which is $R_c^{[j,i]} = \log_2 \left(1 + \mathrm{SINR}_c^{[j,i]}\right)$. Here, in a similar way, the SINR for the common message $W_c^{[i]}$ is defined as

$$\mathrm{SINR}_c^{[j,i]} = \frac{\left| \mathbf{w}_c^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F} \mathbf{H}^{[R,i]} \mathbf{v}_c^{[i]} \right|^2}{I_c^{[j,i]} + N_c^{[j,i]}} \tag{3}$$

where $I_c^{[j,i]} = \sum_{k \neq j} \sum_{l} \left| \mathbf{w}_c^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F} \mathbf{H}^{[R,k]} \mathbf{v}_p^{[l,k]} \right|^2 + \sum_{k \notin \{j,i\}} \left| \mathbf{w}_c^{[j,i]^H} \mathbf{H}^{[j,R]} \mathbf{F} \mathbf{H}^{[R,k]} \mathbf{v}_c^{[k]} \right|^2$.

Now, the achievable DOF for each message is expressed as

$$\eta_p^{[j,i]} = \lim_{P \to \infty} \frac{R_p^{[j,i]}}{\log_2 P} \text{ and } \eta_c^{[i]} = \lim_{P \to \infty} \frac{R_c^{[i]}}{\log_2 P}.$$

Then, the sum of the achievable DOF for K-user Y channels with heterogeneous messages is defined as

$$\eta_{sum}(K) = \sum_{i=1}^{K} \left(\sum_{j \neq i}^{K} \eta_p^{[j,i]} + \eta_c^{[i]} \right).$$

III. Achievable Degree of Freedom with Heterogeneous Messages

In this section, we establish the achievable DoF for the K-user MIMO Y channel with heterogenous messages. We devote this section to proving the following theorem.

Theorem 1: For the heterogeneous message setting of K-user Y channels where each user wishes to obtain the DOF of 1 for each message (i.e., $\eta_p^{[j,i]} = \eta_p = 1,\ \forall j \neq i$ and $\eta_c^{[i]} = \eta_c = 1,\ \forall i$), the total DOF $\eta_{sum}(K) = K^2$ is achieved if the relay has $N = \frac{1}{2}(K-1)(K+2)$ antennas and each user has $M = N - \lfloor \frac{K-1}{2} \rfloor$ antennas.

Proof: Let us first consider $K = 3$, $M = 4$, and $N = 5$ case to explain the basic idea of an achievable scheme. Then, we will generalize this construction to a more general number of users later.

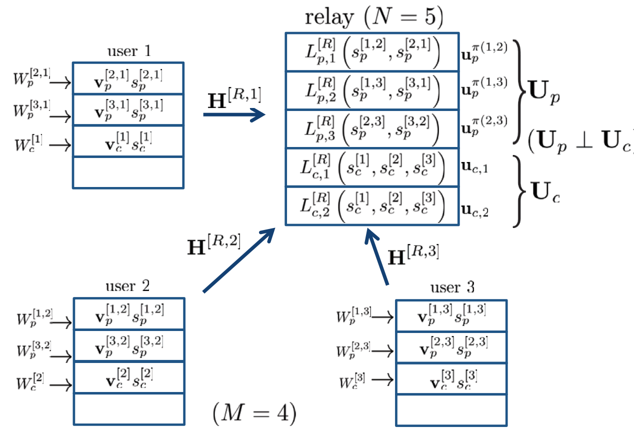

Fig. 1. Signaling method for $K = 3$ in the MAC phase.

A. Three user case $(K = 3)$

During the MAC phase, each user sends two private and one common messages to the relay. The relay receives a total of nine independent messages which consist of six private messages and three common messages. Since the relay has $N = 5$ antennas, each user cooperatively transmits those messages so that they maximize the utility of the signal dimension at the relay. Unlike the previous works, the key idea of the signal dimension usage at the relay is to divide the five signal dimensions into two orthogonal matrices $\mathbf{U}_p$ and $\mathbf{U}_c$ where $\mathbf{U}_p \in \mathbb{C}^{5\times3}$ for the six private messages and $\mathbf{U}_c \in \mathbb{C}^{5\times2}$ for the common messages, as described in Fig. 1.

1) MAC phase transmission: By using the notion of SSA-NC [2] [3], each user designs the beamforming direction so that two desired signals are aligned within the same spatial dimension, i.e., $\mathrm{span}\left(\mathbf{H}^{[R,i]}\mathbf{v}_p^{[j,i]}\right) \doteq \mathrm{span}\left(\mathbf{H}^{[R,j]}\mathbf{v}_p^{[i,j]}\right)$. Specifically, we first choose the three signal dimensions for the private messages as $\mathbf{U}_p = \left[\mathbf{u}_p^{\pi(1,2)} \mathbf{u}_p^{\pi(1,3)} \mathbf{u}_p^{\pi(2,3)}\right]$ where $\mathbf{u}_p^{\pi(i,j)}$ is an unit vector in the intersection subspace between the channel of user i and user j with the index $\pi(i,j)$. Also, the beamforming vector $\mathbf{v}_p^{[j,i]}$ is composed of the power allocation parameter $\alpha_p^{[j,i]}$ and the direction vector $\tilde{\mathbf{v}}_p^{[j,i]}$, i.e., $\mathbf{v}_p^{[j,i]} = \alpha_p^{[j,i]} \tilde{\mathbf{v}}_p^{[j,i]}$.

Then, by solving the following linear equation

$$\begin{bmatrix} \mathbf{I}_N & -\mathbf{H}^{[R,i]} & \mathbf{0} \\ \mathbf{I}_N & \mathbf{0} & -\mathbf{H}^{[R,j]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_p^{\pi(i,j)} \\ \tilde{\mathbf{v}}_p^{[j,i]} \\ \tilde{\mathbf{v}}_p^{[i,j]} \end{bmatrix} = \mathbf{0}, \tag{4}$$

we construct the beamforming vectors $\tilde{\mathbf{v}}_p^{[j,i]}$ and $\tilde{\mathbf{v}}_p^{[i,j]}$ so that the message symbols $s_p^{[j,i]}$ and $s_p^{[i,j]}$ arrive at the signal space $\mathbf{u}_p^{\pi(i,j)}$. Because all channel elements are independently drawn from a continuous distribution, the rank of the nullspace of the matrix in (4) whose size is 10×13 becomes 3, and thus $\mathbf{u}_p^{\pi(i,j)}$ exists with probability one. In addition, the aligned unit vectors $\mathbf{u}_p^{\pi(1,2)}$, $\mathbf{u}_p^{\pi(1,3)}$ and $\mathbf{u}_p^{\pi(2,3)}$ are linearly independent with probability one. Employing the SSA-NC, each user is able to contain two independent private data symbols into one signal dimension for network coding at the relay. This makes a total of six independent private messages span the subspace of $\mathbf{U}_p \in \mathbb{C}^{5\times3}$.

The beamforming vectors for common messages, $\mathbf{v}_c^{[i]}$, are

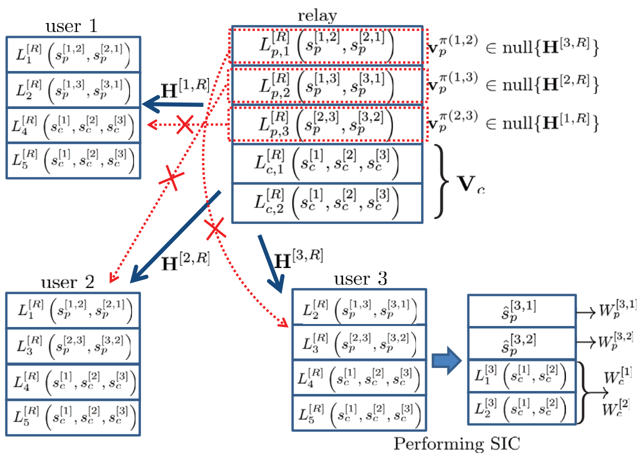

Fig. 2. Signaling method for $K = 3$ in the BC phase.

designed to span the subspace of $\mathbf{U}_c \in \mathbb{C}^{5\times 2}$, which is orthogonal with $\mathbf{U}_p \in \mathbb{C}^{5\times 3}$ for private messages. Therefore, for given $\mathbf{U}_p$, the precoding vector $\mathbf{v}_c^{[i]}$ should satisfy $\text{span}(\mathbf{H}^{[R,i]}\mathbf{v}_c^{[i]}) \not\subset \text{span}(\mathbf{U}_p)$, which is equivalently obtained as

$$\mathbf{v}_c^{[i]} \in \text{null}\left(\mathbf{U}_p^H \mathbf{H}^{[R,i]}\right). \tag{5}$$

Then, the common message $s_c^{[i]}$ arrives at the relay's remaining two column vectors $\mathbf{U}_c = [\mathbf{u}_{c,1}\ \mathbf{u}_{c,2}]$ as $\mathbf{H}^{[R,i]}\mathbf{v}_c^{[i]}s_c^{[i]} = \left(\alpha_{c,1}^{[i]}\mathbf{u}_{c,1} + \alpha_{c,2}^{[i]}\mathbf{u}_{c,2}\right)s_c^{[i]}$ where $\alpha_{c,k}^{[i]}$ is the power allocation parameter. Note that $\mathbf{U}_c$ lies in the orthogonal subspace of $\mathbf{U}_p$, i.e., $\text{span}\left(\mathbf{U}_c\right) \perp \text{span}\left(\mathbf{U}_p\right)$. As a result, the received signal at the relay in (1) can be expressed as

$$\begin{aligned}\mathbf{y}^{[R]} &= \sum_{i=1}^{3}\mathbf{H}^{[R,i]}\left(\sum_{j\neq i}\mathbf{v}_p^{[j,i]}s_p^{[j,i]} + \mathbf{v}_c^{[i]}s_c^{[i]}\right) + \mathbf{n}^{[R]} \\ &= [\mathbf{U}_p\ \mathbf{U}_c]\begin{bmatrix}\mathbf{s}_p^{[R]}\\ \mathbf{s}_c^{[R]}\end{bmatrix} + \mathbf{n}^{[R]}\end{aligned}$$

where we define $\mathbf{s}_p^{[R]}$ and $\mathbf{s}_c^{[R]}$ as $\mathbf{s}_p^{[R]} = [L_{p,1}^{[R]}(s_p^{[1,2]}, s_p^{[2,1]}),\ L_{p,2}^{[R]}(s_p^{[1,3]}, s_p^{[3,1]}),\ L_{p,3}^{[R]}(s_p^{[2,3]}, s_p^{[3,2]})]^T$ and $\mathbf{s}_c^{[R]} = [L_{c,1}^{[R]}(s_c^{[1]}, s_c^{[2]}, s_c^{[3]}),\ L_{c,2}^{[R]}(s_c^{[1]}, s_c^{[2]}, s_c^{[3]})]^T$, respectively, and $L_{p,m}^{[R]}(s_p^{[i,j]}, s_p^{[j,i]}) = \alpha_p^{[j,i]}s_p^{[j,i]} + \alpha_p^{[i,j]}s_p^{[i,j]}$ and $L_{c,m}^{[R]}\ (s_c^{[1]}, s_c^{[2]}, s_c^{[3]}) = \sum_i \alpha_{c,m}^{[i]}s_c^{[i]}$ are the m-th linear combination of the transmitted symbols at the relay with the power normalizing coefficients. Note that $L_{p,m}^{[R]}(s_p^{[i,j]}, s_p^{[j,i]})$ is computed by employing the zero-forcing vector $\mathbf{w}_{p,m}^{\pi(i,j)} \in \text{null}\left((\tilde{\mathbf{U}}_p^{\pi(i,j)})^H\right)$ where $\tilde{\mathbf{U}}_p^{\pi(i,j)}$ is the matrix which excludes the vector $\mathbf{u}_p^{\pi(i,j)}$ from the matrix $\mathbf{U}_p$. The relay can also get the common message symbol vector $\mathbf{s}_c^{[R]}$ with the zero-forcing matrix $\mathbf{W}_c = \mathbf{U}_c$ by canceling the private message signals.

2) BC phase transmission: The main object for designing the relay transmit beamforming vectors is that each user receives two private messages and two common messages without interference as illustrated in Fig. 2. For the private messages, we need to eliminate the interference signal so that each user does not receive the undesired equation symbols as illustrated in the dotted line in Fig. 2.

For example, $L_{p,3}^{[R]}(s_p^{[3,2]}, s_p^{[2,3]})$ is the undesired equation to user 1 while it contains the desired data symbols for user 2 and user 3. By using the fact that $\mathbf{H}^{[1,R]}$ has a size of 4×5 and the entries are drawn from a continuous distribution, we can choose the beamforming vector $\mathbf{v}_p^{\pi(2,3)}$ carrying $L_{p,3}^{[R]}(s_p^{[3,2]}, s_p^{[2,3]})$ so that it lies in the null space of $\mathbf{H}^{[1,R]}$, i.e., $\mathbf{v}_p^{\pi(2,3)} \in \text{null}(\mathbf{H}^{[1,R]})$. Similarly, we construct the beamforming vectors for the other users as $\mathbf{v}_p^{\pi(1,3)} \in \text{null}(\mathbf{H}^{[2,R]})$ and $\mathbf{v}_p^{\pi(1,2)} \in \text{null}(\mathbf{H}^{[3,R]})$. Contrast to the private messages, for the common messages, all users desire to receive all of them. Hence, the beamforming vectors $\mathbf{v}_{c,1}^{[R]}$ and $\mathbf{v}_{c,2}^{[R]}$ carrying the two equations containing the common message symbols $L_{c,1}^{[R]}(s_c^{[1]}, s_c^{[2]}, s_c^{[3]})$ and $L_{c,2}^{[R]}(s_c^{[1]}, s_c^{[2]}, s_c^{[3]})$ can be randomly designed. For instance, we can pick $\mathbf{v}_{c,1}^{[R]} = [0,0,0,1,0]^T$ and $\mathbf{v}_{c,2}^{[R]} = [0,0,0,0,1]^T$. Finally, for given $\mathbf{W}^{[R]} = [\mathbf{w}_p^{\pi(1,2)}, \mathbf{w}_p^{\pi(1,3)}, \mathbf{w}_p^{\pi(2,3)}, \mathbf{W}_c]$ and $\mathbf{V}^{[R]} = [\mathbf{v}_p^{\pi(1,2)}, \mathbf{v}_p^{\pi(1,3)}, \mathbf{v}_p^{\pi(2,3)}, \mathbf{v}_{c,1}, \mathbf{v}_{c,2}]$, the relay beamforming vector $\bar{\mathbf{F}}$ is obtained as $\bar{\mathbf{F}} = \mathbf{V}^{[R]}\mathbf{W}^{[R]^H}$.

3) Decodability: From the relay transmission during the BC phase, employing the zero-forcing (ZF) decoder at the receiver, each user can obtain two equations for the private messages and two equations for the common messages. For example, after applying the ZF decoder such as $\mathbf{W}^{[1]} = \left(\mathbf{H}^{[1,R]}\tilde{\mathbf{V}}_{p,\pi(2,3)}^{[R]}\right)^{-1}$ where $\tilde{\mathbf{V}}_{p,\pi(i,j)}^{[R]}$ denotes the matrix which excludes the vector $\mathbf{v}_p^{\pi(i,j)}$ from the matrix $\mathbf{V}^{[R]}$, user 1 obtains $L_{p,1}^{[R]}(s_p^{[1,2]}, s_p^{[2,1]})$ and $L_{p,2}^{[R]}(s_p^{[1,3]}, s_p^{[3,1]})$ for the private messages and $L_{c,1}^{[R]}(s_c^{[1]}, s_c^{[2]}, s_c^{[3]})$ and $L_{c,2}^{[R]}(s_c^{[1]}, s_c^{[2]}, s_c^{[3]})$ for the common messages.

After self-interference cancelation of $\{s_p^{[2,1]}, s_p^{[3,1]}, s_c^{[1]}\}$, user 1 gets the two desired private messages coming from user 2 and user 3, i.e., $s^{[1,2]}$ and $s^{[1,3]}$, and the two equations $L_{c,1}^{[R]}(s_c^{[2]}, s_c^{[3]})$ and $L_{c,2}^{[R]}(s_c^{[2]}, s_c^{[3]})$. Since two equations $L_{c,1}^{[R]}(s_c^{[2]}, s_c^{[3]})$ and $L_{c,2}^{[R]}(s_c^{[2]}, s_c^{[3]})$ are linearly independent, user 1 can also decode the two common message $s_c^{[2]}$ and $s_c^{[3]}$ by using ZF decoder. User 1 finally obtains two private messages and two common messages from user 2 and user 3. By symmetry, user 2 and user 3 also can decode two private messages and two common messages. This implies that user i can construct the receive combining vectors $\mathbf{w}_p^{[i,j]}$ and $\mathbf{w}_c^{[i,j]}$ which make the interference terms $I_p^{[i,j]}$ in (2) and $I_c^{[i,j]}$ in (3) perfectly eliminated. Consequently, the DOF of $\eta_p^{[i,j]} = 1$ and $\eta_c^{[i]} = 1$ is achieved, which leads to achieve the total DOF $\eta_{sum}(3) = 9$ when $M = 4$ and $N = 5$ with $K = 3$.

B. Extension to the general K-user case

In this subsection, we generalize the proposed scheme to the general K-user case, assuming that each user achieves the DOF of 1 for each message ($\eta_p = \eta_c = 1$).

1) MAC phase: Since each user transmits total K messages ($K-1$ private messages and one common message), the relay receives K^2 messages during the MAC phase. As explained in the previous subsection, the relay needs $\frac{K(K-1)}{2}$ dimensional signal space for containing the private messages and $K-1$ dimensional signal space for the common messages, which requires $N = \frac{K(K-1)}{2} + (K-1) = \frac{1}{2}(K-1)(K+2)$ antennas

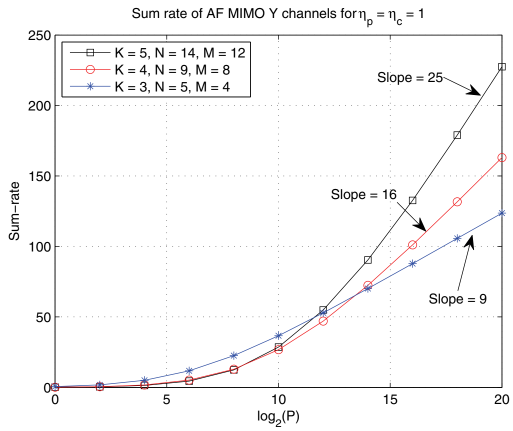

Fig. 3. Sum rate performance for $K = 3$, 4 and 5.

for the relay to properly contain the K^2 messages during the MAC phase.

In order to properly divide the signal space for private and common messages, the beamforming vectors $\mathbf{v}_p^{[j,i]}$ and $\mathbf{v}_c^{[i]}$ should be designed by satisfying the condition of (4) and (5), respectively. Since the size of matrix in (4) is $2N \times (N+2M)$ and its entries are chosen in a continuous distribution, a solution of (4) exists if $M > \frac{N}{2} = \frac{1}{4}(K-1)(K+2)$ for the beamforming vectors of user i and user j and the aligned channel vector $\mathbf{u}_p^{\pi(i,j)}$. Given the $\frac{K(K-1)}{2}$ aligned vectors $\mathbf{u}_p^{\pi(i,j)}$, $\forall i \neq j$, i.e., $\mathbf{U}_p \in \mathbb{C}^{N \times \frac{K(K-1)}{2}}$, a solution for the precoding vector $\mathbf{v}_c^{[i]}$ in (5) exists almost surely if $M > \frac{K(K-1)}{2}$. As a result, if each user has $M = N - \lfloor \frac{K-1}{2} \rfloor$ antennas, the conditions $M > \max\{\frac{1}{4}(K-1)(K+2), \frac{K(K-1)}{2}\}$ are met so that it is possible to perform the proposed scheme during the MAC phase.

2) BC phase: Recall that while $K-1$ common message symbols are the desired signals for all users, the private message pairs excluding the desired $K-1$ messages among $\frac{K(K-1)}{2}$ private message pairs become interference signals. The main idea of the relay beamforming is to eliminate inter-user interference signals. To accomplish this, we use the asymmetry of the number of antennas between the relay and users, which generates null space. Since there are $\lfloor \frac{K-1}{2} \rfloor$ nullity in the downlink channel $\mathbf{H}^{[i,R]}$ if $M = N - \lfloor \frac{K-1}{2} \rfloor$, the relay selects $\lfloor \frac{K-1}{2} \rfloor$ private message equations $L_{p,m}^{[R]}(s_p^{[i,j]}, s_p^{[j,i]})$ which are interference to user k and designs their beamforming vectors $\mathbf{v}_p^{\pi(i,j)}$ as $\mathbf{v}_p^{\pi(i,j)} = \text{null}\{\mathbf{H}^{[k,R]}\}$. The relay can properly choose $K\lfloor \frac{K-1}{2} \rfloor$ symbols and compute the beamforming vectors so that each user does not receive $\lfloor \frac{K-1}{2} \rfloor$ interference symbols among a total of $\frac{K(K-1)}{2}$ private message pairs.

For the remaining $K\left(\frac{K-1}{2} - \lfloor \frac{K-1}{2} \rfloor\right)$ private message pairs, the relay arbitrarily computes the beamforming vectors which are linearly independent with the predetermined beamforming vectors. Contrast to the beamforming vectors for private messages, the relay arbitrary chooses $K-1$ independent vectors $\mathbf{v}_{c,m}$ for the common messages equation $L_{c,m}^{[R]}(\{s_c^{[i]}, \forall i\})$ for $m = 1, \cdots, K-1$. Then each user receives $N - \lfloor \frac{K-1}{2} \rfloor$ network coded linear equations during the BC phase.

3) Decodability: Finally, each user tries to decode $K-1$ common messages and $K-1$ private messages from its received signals. Since $N - \lfloor \frac{K-1}{2} \rfloor$ symbols arrive at each user and each user has enough antennas $M = N - \lfloor \frac{K-1}{2} \rfloor$, the received message symbols can be resolved without any interference by applying the ZF decoder. After user i gets $K-1$ private message equations and $K-1$ common message equations, self-interference cancelation is performed using the side information $\{s_p^{[1,i]}, \cdots, s_p^{[K,i]}, s_c^{[i]}\}$ and the desired private and common messages are decoded. In other words, user i can obtain the receive vectors $\mathbf{w}_p^{[i,j]}$ and $\mathbf{w}_c^{[i,j]}$ to cancel the interference terms $I_p^{[i,j]}$ and $I_c^{[i,j]}$ in (2) and (3), respectively.

As a result, all users achieve the DOF of 1 for each message ($\eta_p = \eta_c = 1$), as they are able to transmit $K-1$ private messages and one common message to the desired users without any interference. Thus, the proposed signaling method achieves the DOF of $\eta_{sum}(K) = K(K-1) + K = K^2$ with the antenna configuration of $N = \frac{1}{2}(K-1)(K+2)$ and $M = N - \lfloor \frac{K-1}{2} \rfloor$. ■

Remark 1: By using the proposed scheme, the result of Theorem 1 can be easily extended to the general case of η_p and η_c as follows: For K-user Y channels with $\eta_p \geq 1$ and $\eta_c \geq 1$, the total DOF $\eta_{sum} = K(K-1)\eta_p + K\eta_c$ is achieved if the relay has $N = \frac{\eta_p K(K-1)}{2} + \eta_c(K-1)$ antennas and all users have $M = N - \lfloor \frac{\eta_p(K-1)}{2} \rfloor$ antennas.

C. Simulation Results

We provide the simulation results to evaluate the sum rate performance of the proposed scheme. From the simulation results, we will confirm the feasibility condition in Theorem 1. For simplicity, we assume that each beamforming vector is randomly chosen satisfying the SSA-NC condition and each user allocates equal power to each beamforming vector, i.e., $\|\mathbf{v}_p^{[j,i]}\|^2 = \|\mathbf{v}_c^{[i]}\|^2 = \frac{P}{K}$ for $\forall i, j, m$.

Fig. 3. exhibits the sum rate performance of $K = 3$, 4 and 5 with $\eta_p = \eta_c = 1$ with respect to $\log_2(P)$. The slope of the sum rate curves indicates the achievable DOF. For the case of $K = 3$, the slope at high P becomes 9 when $M = 4$ and $N = 5$. Also, the slope is 16 for $K = 4$, $N = 9$ and $M = 8$ and is 25 for $K = 5$, $N = 14$ and $M = 12$ as expected. It is observed that the proposed scheme achieves the derived DOF of K^2 when the feasibility condition is satisfied.

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