Precoding Designs Based on Minimum Distance for Two-Way Relaying MIMO Systems with Physical Network Coding

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***Abstract*—In this paper, we propose new precoding methods for two-way multiple input multiple output physical network coding (PNC) systems which employ the modulo operation. In our work, the transmit and receive filters are determined to maximize the minimum distance of the received constellations assuming global channel state information. The precoding operations are separately optimized for the multiple access (MA) and the broadcast stages, and the optimal precoding is obtained by applying a semidefinite relaxation method. Especially, we prove that for the system with linear detection the modulo operation for the PNC achieves optimality with the derived precoding for the MA stage in terms of the minimum distance. Also, we present a closed-form solution for the optimal filter designs for two special cases. For computing solutions, we transform our max min problem into a simple maximization problem by imposing additional constraints. Also, we propose a suboptimal non-iterative precoding scheme whose performance is within 1 dB at a bit error rate (BER) of 10^{-4} compared to the optimum iterative method with much reduced complexity. Finally, the simulation results show that the proposed systems achieve 2-3 dB gains at a BER of 10^{-4} compared to the optimal amplify-and-forward systems.**

***Index Terms*—Physical network coding, two-way Relay, minimum distance.**

I. INTRODUCTION

OVER the past years, relaying transmission has been a subject of intense research for extending coverage or increasing the system capacity [2] [3]. Most relay systems are assumed to operate in the half-duplex mode where a relay node (RN) does not transmit and receive signals simultaneously. Thus, they suffer from a spectral efficiency loss due to the $\frac{1}{2}$ pre-log factor in one-way systems [3]. In order to overcome the loss in the one-way system, two-way half-duplex systems have been proposed in [4] and [5]. Some theoretical results on the relaying systems have been investigated in [6] and [7].

There are two popular protocols for relaying transmission. One is amplify-and-forward (AF) [6] [8] [9] where the RN just amplifies the received signal, and the other is decode-and-forward (DF) where the RN performs decoding. In two-way AF systems, two end nodes (EN) simultaneously transmit their information to the RN at the first time slot, and the RN broadcasts the mixed information to ENs at the second time slot. Since each EN knows its own transmitted data, self-interference in the transmitted signals can be removed from the received signal. This process is also referred to as analogue network coding. In contrast, for two-way DF systems, it is more natural for ENs to transmit their information in different time slots as in [10] and [11]. Subsequently, the RN encodes the decoded information with network coding [12], and the desired information can be extracted by utilizing the network coding at the ENs.

Recently, denoise-and-forward (DNF) with physical network coding (PNC) for two-way relaying networks was introduced in [13]–[19]. In this protocol, after the ENs transmit their data simultaneously, the RN just detects the symbols instead of decoding the information bits. Especially, the RN employs the PNC which suppresses detection errors caused by interference of the multiple access (MA) stage. If the RN jointly decodes the information from both ENs as in DF, the system performance may suffer from the MA interference as shown in [13]. In DNF systems, the PNC plays an important role to overcome the MA interference. Assuming perfect synchronization for both channels of ENs, we can apply an efficient PNC scheme based on the modulo operation [14] to remove the MA interference completely. Alternatively, using an adaptive PNC design according to instantaneous channel conditions, the MA interference would be suppressed as shown in [15].

The combination of the DNF scheme with multiple antennas has been studied in [20], in which precoding techniques are proposed when adaptive network coding or a modulo operation is utilized for the PNC. In [20], the precoding of the ENs was designed for several different scenarios depending on available channel state information (CSI). For the case of global CSI at the RN and the EN, their work first chooses the modulo operation for PNC, and then the precoding is optimized in

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terms of the minimum distance by using a Lagrange method. In their work, there are some limitations. First, the system in [20] employs the modulo operation without any proof of the optimality for PNC. Also, the number of Lagrange multipliers increases exponentially with the cardinality of the employed modulation set. For example, when 16-QAM is used at both ENs for the scheme in [20], there are more than 65536 Lagrange multipliers to be determined. Thus, it would be prohibitive to solve the problem at every channel realizations. In our paper, we attempt to address a solution which can avoid these problems.

First, we propose a new precoding method for single stream multiple-input multiple-output (MIMO) systems with a PNC scheme based on the modulo operation [14]. In our work, we focus on the minimum distance of the received constellations which is associated with the symbol error rate (SER) [21]. We separately optimize the filters for the MA and BC stages to maximize the respective minimum distance. Then, the transmit and receive filters of the ENs and the RN are optimized for the MA and BC stages assuming global CSI. For our work, we approach the problem by examining an upper bound of the minimum distance. The precoding can be obtained to maximize the upper bound by applying a semidefinite relaxation method [22]. We prove that for the system with linear detection, the precoding of the MA stage combined with the modulo operation for PNC is jointly optimal.

In terms of detection at the RN, we consider both maximum likelihood detection (MLD) and linear detection (LD). For the LD system, an iterative method is proposed to obtain the optimal precoding which maximizes the minimum distance. Also, we derive closed-form precoding methods of the optimal transmit and receive filters of the LD systems for certain configurations. Especially, in order to find the exact solutions, our max min problem can be changed into a simple maximization problem by imposing additional constraints. Also, to reduce the complexity, a suboptimal non-iterative precoding scheme is presented. Simulation results show that the performance of the suboptimal strategy for LD systems is within 1 dB at a bit error rate (BER) of 10^{-4} compared to the optimal performance. Also, the proposed systems achieve 2-3 dB gains at a BER of 10^{-4} over the optimal AF systems.

Throughout this paper, normal letters represent scalar quantities, boldface letters indicate vectors and boldface uppercase letters designate matrices. $\mathbb{C}$ denotes a set of complex numbers, and $\mathbf{A}^H$ and $\mathrm{tr}(\mathbf{A})$ stand for Hermitian and trace of a matrix $\mathbf{A}$, respectively. Also, a^* indicates the conjugate of a complex number a, and $||\cdot||$ is defined by Frobenius norm. Finally, $\mathbf{I}_a$ is denoted as an identity matrix of size a.

II. SYSTEM MODEL

Fig. 1 illustrates a relay system where two ENs A and B are equipped with N_A and N_B antennas, respectively, and one RN has N_R antennas. Both ENs exchange the data through the RN assuming the same modulation level for the data symbols. We assume that global CSI is available at all nodes. Also, there is no direct link between the ENs, and both ENs are assumed to transmit a single stream. Half duplex systems are assumed where transmission and reception at a certain node must be carried out in different time slots.

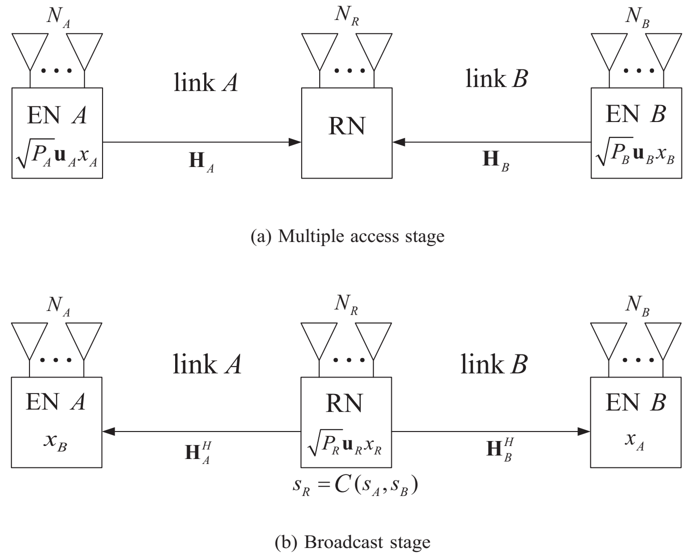


Fig. 1. Schematic diagram of a precoding scheme with PNC

Let us define x_i as one of an M-QAM symbol transmitted by EN i for $i = A$ and B, which consists of the inphase x_{iI} and quadrature x_{iQ} as $x_i = x_{iI} + jx_{iQ}$ ($j = \sqrt{-1}$) with $E\{|x_i|^2\} = 1$. Here, x_{ik} for $k = I$ and Q is obtained as $x_{ik} = \mathcal{M}(s_{ik})$ where $\mathcal{M}(\cdot)$ denotes a $\sqrt{M}$-PAM constellation mapper, and s_{iI} and $s_{iQ} \in \{0, 1, ..., \sqrt{M} - 1\}$ indicate the equally probable information corresponding to the inphase and quadrature of x_i, respectively. Then, x_{ik} is calculated as

$$x_{ik} = \frac{2s_{ik} - (\sqrt{M} - 1)}{\sqrt{2\mathcal{E}}} \tag{1}$$

where $\mathcal{E} = (M - 1)/3$. For BPSK, x_{iI} is defined as $x_{iI} = 2s_{iI} - 1$ where $s_{iI} \in \{0,\ 1\}$.

At the MA stage, both symbols at EN A and B are precoded by the transmit filters $\mathbf{u}_A \in \mathbb{C}^{N_A \times 1}$ and $\mathbf{u}_B \in \mathbb{C}^{N_B \times 1}$ with $||\mathbf{u}_A||^2 = ||\mathbf{u}_B||^2 = 1$, respectively, and are transmitted simultaneously to the RN. Fig. 1 (a) depicts the schematic diagram for the MA stage where link i indicates the link between EN i and the RN for $i = A$ and B. Then, the received signal at the RN is expressed as

$$\mathbf{y}_R = \sqrt{P_A}\mathbf{H}_A\mathbf{u}_A x_A + \sqrt{P_B}\mathbf{H}_B\mathbf{u}_B x_B + \mathbf{z}_R \tag{2}$$

where P_i and $\mathbf{H}_i \in \mathbb{C}^{N_R \times N_i}$ denote the transmitted power and the channel matrix of link i, respectively, for $i = A$ and B, and $\mathbf{z}_R \sim \mathcal{N}(0, \sigma_R^2\mathbf{I}_{N_R})$ represents the circularly symmetric complex Gaussian noise. We also assume that the elements of the channels $\mathbf{H}_A$ and $\mathbf{H}_B$ are generated with an independent and identically distributed (i.i.d.) complex Gaussian distribution with zero mean and unit variance, whose magnitudes have a Rayleigh fading distribution.

After receiving the signal from the ENs, the RN first detects $(\widehat{x}_A, \widehat{x}_B)$, and then employs PNC to determine the transmitted symbol s_R from the estimated symbol set $(\widehat{x}_A, \widehat{x}_B)$, where s_R is defined as the transmitted data at the RN. For detecting two symbols $(\widehat{x}_A, \widehat{x}_B)$ at the RN, we consider both MLD and LD. For the MLD, the following operation is utilized.

$$(\widehat{x}_A, \widehat{x}_B) = \underset{x_A, x_B}{\mathrm{argmax}} ||\mathbf{y}_R - \sqrt{P_A}\mathbf{H}_A\mathbf{u}_A x_A - \sqrt{P_B}\mathbf{H}_B\mathbf{u}_B x_B||^2. \tag{3}$$

The detection operation for LD will be described later.

After the detection, the RN generates s_{Rk} by the modulo operation [14] as

$$\begin{aligned} s_{Rk} &= C_k(s_{Ak},\ s_{Bk}) \\ &= (s_{Ak}+s_{Bk}) \bmod \sqrt{M} \qquad \text{for } k = I \text{ and } Q \end{aligned} \tag{4}$$

where $C_k(\cdot,\cdot)$ indicates the PNC function for inphase or quadrature, and $(\cdot) \bmod \sqrt{M}$ is defined by the modulo operation of size $\sqrt{M}$. In (4), the modulo operation separately computes the inphase and the quadrature. For BPSK, s_{RI} is given by $s_{RI} = C_I(s_{AI},\ s_{BI}) = (s_{AI}+s_{BI}) \bmod 2$. Note that the candidate set of (x_A, x_B) for the detection (3) is changed as $\{(s_{AI}, s'_{AI}, s_{BI}, s'_{BI})|\ C_I(s_{AI}, s_{BI}) \neq C_I(s'_{AI}, s'_{BI})$ or $C_Q(s_{AQ}, s_{BQ}) \neq C_Q(s'_{AQ}, s'_{BQ})\}$ by the PNC function.

At the broadcast (BC) stage described in Fig. 1 (b), the RN broadcasts the determined symbol $x_R = \mathcal{M}(s_R)$ to EN A and B. Then, the received signals at the ENs can be expressed as

$$\mathbf{y}_i = \sqrt{P_R}\mathbf{H}_i^H\mathbf{u}_R x_R + \mathbf{z}_i \qquad \text{for } i = A \text{ and } B$$

where P_R denotes the transmitted power of the RN, $\mathbf{z}_i$ equals the i.i.d. Gaussian noise with variance σ_i^2, and $\mathbf{u}_R \in \mathbb{C}^{N_R\times 1}$ represents the beamforming vector of the RN with $||\mathbf{u}_R||^2 = 1$. For simplicity, we assume reciprocal channels for both stages (i.e. the channel of the BC stage is equal to the Hermitian channel of the MA stage).

At EN i, the linear combining vector $\mathbf{g}_i \in \mathbb{C}^{N_i\times 1}$ is employed with $||\mathbf{g}_i||^2 = 1$. Although the LD is suboptimal in general, it was shown in [23] that the LD approaches the optimal performance when one stream is transmitted. The received signal with the receive filter $\mathbf{g}_i$ is rewritten by

$$y_i = \sqrt{P_R}\mathbf{g}_i^H\mathbf{H}_i^H\mathbf{u}_R x_R + z_i \qquad \text{for } i = A \text{ and } B \tag{5}$$

where $y_i \triangleq \mathbf{g}_i^H\mathbf{y}_i$ and $z_i \triangleq \mathbf{g}_i^H\mathbf{z}_i$. The filtered noise z_i has σ_i^2 variance. Then, EN A detects the symbol x_R using MLD. After that, utilizing its own symbol s_A, the other EN's symbol s_B is obtained as

$$s_{Bk} = (s_{Ak} + s_{Rk}) \bmod \sqrt{M}.$$

Similarly, EN B can detect the symbol s_A.

III. Proposed Strategies for PNC MIMO Systems

In this section, we present a method to maximize the minimum distance subject to individual power constraints $P_A \leq P_{A_C}$, $P_B \leq P_{B_C}$ and $P_R \leq P_{R_C}$ where P_{A_C}, P_{B_C} and P_{R_C} stand for the power constraints of EN A, B and the RN, respectively. We separately optimize the filters for the MA and BC stages to maximize the respective minimum distance. Especially, for the MA stage, we solve the problem considering the PNC. We start with optimizing the MA stage for MLD and LD in the following.

A. Optimization of the MA Stage for MLD systems

Here, we design the optimal transmit filter at the MA stage when the RN employs MLD. It is well-known that in order to optimize the error performance of the MLD, the minimum distance should be maximized [24]. The squared minimum distance of the MA stage is defined as

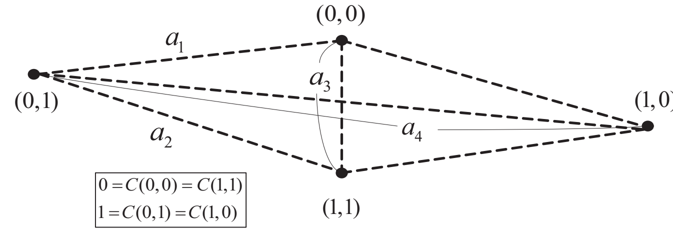


Fig. 2. Received constellation points (s_A, s_B) for two way PNC with BPSK

$$\begin{aligned} d^2_{\min,\text{MA}} \triangleq \min_{\mathbb{S}} \| &\sqrt{P_A}\mathbf{H}_A\mathbf{u}_A\Big(\delta(s_{AI}, s'_{AI}) + j\delta(s_{AQ}, s'_{AQ})\Big) \\ &+ \sqrt{P_B}\mathbf{H}_B\mathbf{u}_B\Big(\delta(s_{BI}, s'_{BI}) + j\delta(s_{BQ}, s'_{BQ})\Big)\|^2/\sigma_R^2. \end{aligned} \tag{6}$$

where $\delta(s_{ik}, s'_{ik}) \triangleq \mathcal{M}(s_{ik}) - \mathcal{M}(s'_{ik})$ for $i \in \{A, B, R\}$ and $k \in \{I, Q\}$, and $\mathbb{S} \triangleq \{(s_{AI}, s'_{AI}, s_{BI}, s'_{BI})|C_I(s_{AI}, s_{BI}) \neq C_I(s'_{AI}, s'_{BI})$ or $C_Q(s_{AQ}, s_{BQ}) \neq C_Q(s'_{AQ}, s'_{BQ})\}$. Then, we consider the following relation [16] about the minimum distance

$$d^2_{\min,\text{MA}} \leq U_{\text{MA}} \triangleq \min\left(\frac{||\sqrt{P_A}\mathbf{H}_A\mathbf{u}_A\delta_{\min}||^2}{\sigma_R^2}, \frac{||\sqrt{P_B}\mathbf{H}_B\mathbf{u}_B\delta_{\min}||^2}{\sigma_R^2}\right) \tag{7}$$

where $\delta_{\min} \triangleq \min|\delta(s_{iI}, s'_{iI})| = \min|\delta(s_{iQ}, s'_{iQ})|$ for $s_{ik} \neq s'_{ik}$. Equation (7) implies that the minimum distance of the MA stage cannot be greater than that of link A or B due to the MA interference.

First, we optimize the transmit filter $\mathbf{u}_i$ to maximize the upper bound U_{MA} in (7). Suppose that singular value decomposition (SVD) of the channel matrices is expressed by $\mathbf{H}_i = \mathbf{U}_i\Sigma_i\mathbf{V}_i^H$ ($i = A$ and B) where $\mathbf{U}_i \in \mathbb{C}^{N_R\times N_R}$ and $\mathbf{V}_i \in \mathbb{C}^{N_i\times N_i}$ are the left and right singular matrices of $\mathbf{H}_i$, respectively, and the diagonal terms of $\Sigma_i \in \mathbb{C}^{N_R\times N_i}$ equal ordered singular values of $\mathbf{H}_i$. Then, using a beamforming solution in [25] to maximize $||\sqrt{P_i}\mathbf{H}_i\mathbf{u}_i\delta_{\min}||$, $\mathbf{u}_i$ can be derived as

$$\mathbf{u}_i^\dagger = \mathbf{v}_{i1} \qquad \text{for } i = A \text{ and } B \tag{8}$$

where $\mathbf{v}_{i1}$ represents the first column of $\mathbf{V}_i$.

Second, we will show that in the BPSK system $d^2_{\min,\text{MA}}$ equals the upper bound U_{MA} by utilizing the modulo operation for the PNC. With BPSK, there are four candidates of the minimum distance : $d_1^2 = ||2\sqrt{P_A}\mathbf{H}_A\mathbf{u}_A^\dagger||^2/\sigma_R^2$ for $s_{AI} = s_{BI} = s'_{BI} = 0$ and $s'_{AI} = 1$, $d_2^2 = ||2\sqrt{P_B}\mathbf{H}_B\mathbf{u}_B^\dagger||^2/\sigma_R^2$ for $s_{AI} = s'_{AI} = s_{BI} = 0$ and $s'_{BI} = 1$, $d_3^2 = ||2\sqrt{P_A}\mathbf{H}_A\mathbf{u}_A^\dagger + 2\sqrt{P_B}\mathbf{H}_B\mathbf{u}_B^\dagger||^2/\sigma_R^2$ for $s_{AI} = s_{BI} = 0$ and $s'_{AI} = s'_{BI} = 1$, and $d_4^2 = ||2\sqrt{P_A}\mathbf{H}_A\mathbf{u}_A^\dagger - 2\sqrt{P_B}\mathbf{H}_B\mathbf{u}_B^\dagger||^2/\sigma_R^2$ for $s_{AI} = s'_{BI} = 0$ and $s'_{AI} = s_{BI} = 1$.

Then, when choosing the modulo operation for the PNC, we have the relations $C_I(0,1) = C_I(1,0)$ and $C_I(0,0) = C_I(1,1)$. Thus, from these relations and the condition $\mathbb{S}$ in (6), d_3 and d_4 is excluded in the candidate set of the minimum distance. As a result, the minimum distance is given by

$\min(d_1^2, d_2^2)$ which is equal to the upper bound U_{MA} in (7). Note that the minimum distance achieves the maximum upper bound.

Fig. 2 depicts an example of the received constellation points for BPSK, where there are four distances among constellation points. In this figure, it can be shown that the distances a_1 and a_2 are determined by the gains of link A and B assuming no MA interference, respectively, i.e. $a_1^2 = ||\sqrt{P_A}\mathbf{H}_A\mathbf{u}_A^\dagger\delta_{\min}||^2$ and $a_2^2 = ||\sqrt{P_B}\mathbf{H}_B\mathbf{u}_B^\dagger\delta_{\min}||^2$. Then, the other distances a_3 and a_4 can be regarded to be generated by the MA interference. Since a_3 becomes the minimum distance without PNC in this figure, a_3 would dominate the system performance. However, after a_3 and a_4 are excluded in the candidate set of $d_{\min}$ by the modulo operation for PNC, a_3 and a_4 have no effect on the system performance. Note that the MA interference is removed by utilizing the modulo operation for PNC in the BPSK system, so that $d^2_{\min,\text{MA}}$ approaches the upper bound U_{MA}. However, for higher modulation in the ML system, we do not know whether there exists the optimal PNC which achieves the optimum minimum distance. In the following section, we consider general modulation levels utilizing linear detection at the RN.

B. Optimization of the MA Stage for LD systems

In this subsection, we derive the optimal transmit and receive filters for general constellations assuming that the RN employs LD. In the LD system, the receive filter projects $\mathbf{y}_R$ onto one complex dimensional signal by considering that PNC generates one symbol s_R. Denoting the receive filter at the RN as $\mathbf{g}_R \in \mathbb{C}^{N_R\times 1}$ with $||\mathbf{g}_R||^2 = 1$, the received signal in (2) is rewritten as

$$y_R = \sqrt{P_A}\mathbf{g}_R^H\mathbf{H}_A\mathbf{u}_A x_A + \sqrt{P_B}\mathbf{g}_R^H\mathbf{H}_B\mathbf{u}_B x_B + z_R \qquad (9)$$

where $y_R = \mathbf{g}_R^H\mathbf{y}_R$ and $z_R = \mathbf{g}_R^H\mathbf{z}_R$.

To simplify the derivation, we define $\mathbf{u}_i = \mathbf{V}_i\mathbf{f}_i$ and $\mathbf{g}_R = \mathbf{U}_A\mathbf{t}_R$. Since $\mathbf{U}_i$ and $\mathbf{V}_i$ are unitary matrices, we will optimize $\mathbf{f}_i$ and $\mathbf{t}_R$ instead of $\mathbf{u}_i$ and $\mathbf{g}_R$ without loss of generality. With these relations, it follows $y_R = \sqrt{P_A}\mathbf{t}_R^H\Sigma_A\mathbf{f}_A x_A + \sqrt{P_B}\mathbf{t}_R^H\mathbf{U}_R\Sigma_B\mathbf{f}_B x_B + z_R$, where $\mathbf{U}_R \triangleq \mathbf{U}_A^H\mathbf{U}_B$. Then, similar to the MLD system, we have the following bound

$$d^2_{\min,\text{MA LD}} \le U_{\text{MA,LD}} \triangleq \min\left(\frac{|\sqrt{P_A}\mathbf{t}_R^H\Sigma_A\mathbf{f}_A\delta_{\min}|^2}{\sigma_R^2}, \frac{|\sqrt{P_B}\mathbf{t}_R^H\mathbf{U}_R\Sigma_B\mathbf{f}_B\delta_{\min}|^2}{\sigma_R^2}\right). \qquad (10)$$

Note that $U_{\text{MA,LD}}$ is the same as the minimum of the signal-to-noise ratio (SNR) with no MA interference.

Next, we maximize the upper bound $U_{\text{MA,LD}}$. We will show that by adopting the modulo operation for PNC, $d^2_{\min,\text{MA LD}}$ achieves the maximum $U_{\text{MA,LD}}$. First, we maximize the SNR in (10) individually. It can be easily checked that we need $P_A = P_{A_C}$ and $P_B = P_{B_C}$ to maximize the SNRs. Applying the Cauchy-Schwarz inequality, $\mathbf{f}_A$ and $\mathbf{f}_B$ which maximize each SNR can be computed as $\mathbf{f}_A = \frac{\Sigma_A^H\mathbf{t}_R}{\sqrt{\mathbf{t}_R^H\Sigma_A\Sigma_A^H\mathbf{t}_R}}$ and $\mathbf{f}_B = \frac{\Sigma_B^H\mathbf{U}_R^H\mathbf{t}_R}{\sqrt{\mathbf{t}_R^H\mathbf{U}_R\Sigma_B\Sigma_B^H\mathbf{U}_R^H\mathbf{t}_R}}$, respectively.

Substituting these equations into (10), our problem can be formulated as

$$\mathbf{t}_R = \arg\max_{\mathbf{t}_R} \min(\alpha,\ \beta) \quad \text{s.t. } ||\mathbf{t}_R||^2 = 1 \qquad (11)$$

where $\alpha \triangleq \mathbf{t}_R^H\Sigma_A\Sigma_A^H\mathbf{t}_R$ and $\beta \triangleq P_{BA}\mathbf{t}_R^H\mathbf{U}_R\Sigma_B\Sigma_B^H\mathbf{U}_R^H\mathbf{t}_R$. Here, P_{BA} represents P_{B_C}/P_{A_C}. Although this problem is not concave in general, it can be solved by a semidefinite relaxation method [22]. Once $\mathbf{t}_R$ is computed iteratively from convex optimization based on the relaxation method, we can also calculate $\mathbf{g}_R^\dagger$ and $\mathbf{u}_i^\dagger$.

From the optimal $\mathbf{t}_R^\dagger$, the upper bound $U_{\text{MA,LD}} = \min(P_{A_C}\mu,\ P_{B_C}\nu)\delta^2_{\min}/\sigma_R^2$ can be maximized where $\mu \triangleq \mathbf{t}_R^{\dagger H}\Sigma_A\Sigma_A^H\mathbf{t}_R^\dagger$ and $\nu \triangleq \mathbf{t}_R^{\dagger H}\mathbf{U}_R\Sigma_B\Sigma_B^H\mathbf{U}_R^H\mathbf{t}_R^\dagger$ represent the squared effective channels for both links. Then, while keeping the maximum value of $U_{\text{MA,LD}}$, the values of P_A and P_B are recomputed as

$$\begin{aligned} &P_A^\dagger = P_{A_C} \text{ and } P_B^\dagger = \frac{\mu}{\nu}P_{A_C} && \text{if } P_{A_C}\mu \le P_{B_C}\nu \\ &P_A^\dagger = \frac{\nu}{\mu}P_{B_C} \text{ and } P_B^\dagger = P_{B_C} && \text{if } P_{A_C}\mu > P_{B_C}\nu. \end{aligned} \qquad (12)$$

Note that the above expressions maintain the maximum value of $U_{\text{MA,LD}}$. Also, these conditions make both links have the equal effective channel ($P_A^\dagger\mu = P_B^\dagger\nu$), and this equal channel approach will be useful to show that the modulo operation is optimal with the derived precoding.

Now, we prove the joint optimality of the modulo operation together with the above obtained precoding. We have already known that $\mathbf{P}_i^\dagger$, $\mathbf{g}_R^\dagger$ and $\mathbf{u}_i^\dagger$ maximize the upper bound of the minimum distance. Putting these parameters into (9), we will check the actual minimum distance with PNC. Denoting the same effective channel in (12) as $g = \sqrt{P_A^\dagger}\mathbf{g}_R^{\dagger H}\mathbf{H}_A\mathbf{u}_A^\dagger = \sqrt{P_B^\dagger}\mathbf{g}_R^{\dagger H}\mathbf{H}_B\mathbf{u}_B^\dagger$ which is the channel gain in (9), the squared minimum distance in the LD system can be expressed as

$$d^2_{\min,\text{MA LD}} \triangleq \min_{\mathbb{S}} \frac{g^2}{\sigma_R^2}\big|\big|\delta(s_{AI}, s'_{AI}) + j\delta(s_{AQ}, s'_{AQ}) + \delta(s_{BI}, s'_{BI}) + j\delta(s_{BQ}, s'_{BQ})\big|\big|^2. \qquad (13)$$

The equation (13) can be constructed by applying the LD receiver $\mathbf{g}_R^\dagger$ to (6).

By substituting (1) into (13), it follows

$$d^2_{\min,\text{MA LD}} = \min_{\mathbb{S}} \frac{2g^2}{\mathcal{E}\sigma_R^2}\ \big|\big|Z_I + jZ_Q\big|\big|^2, \qquad (14)$$

where $Z_I \triangleq (s_{AI} + s_{BI}) - (s'_{AI} + s'_{BI})$ and $Z_Q \triangleq (s_{AQ} + s_{BQ}) - (s'_{AQ} + s'_{BQ})$. In (14), both Z_I and Z_Q are integer numbers from the interval $[-(\sqrt{M}-1)\ \ \sqrt{M}-1]$ since we have $s_{ik} \in \{0, 1, \cdots, \sqrt{M}-1\}$. Note that if the condition set $\mathbb{S} = \{(s_{AI}, s'_{AI}, s_{BI}, s'_{BI})|C_I(s_{AI}, s_{BI}) \neq C_I(s'_{AI}, s'_{BI})$ or $C_Q(s_{AQ}, s_{BQ}) \neq C_Q(s'_{AQ}, s'_{BQ})\}$ is not considered in (14), the minimum distance would be zero. For example, substituting $s_{AI} = s_{AQ} = s'_{BI} = s'_{BQ} = 0$ and $s_{BI} = s_{BQ} = s'_{AI} = s'_{AQ} = 1$ into Z_I and Z_Q, it follows $Z_I = Z_Q = 0$, and then, we can check $d^2_{\min,\text{MA LD}} = 0$ in (14).

Let us employ the modulo operation for PNC to see how the zero value of the minimum distance is changed. The

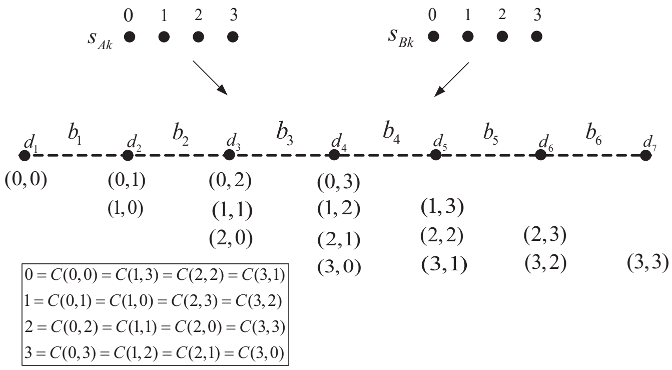


Fig. 3. Received constellation points (s_{Ak}, s_{Bk}) for two way PNC with 4-PAM

necessary and sufficient condition for zero minimum distance in (14) is $s_{AI} + s_{BI} = s'_{AI} + s'_{BI}$ and $s_{AQ} + s_{BQ} = s'_{AQ} + s'_{BQ}$. This condition leads to both $C_I(s_{AI}, s_{BI}) = C_I(s'_{AI}, s'_{BI})$ and $C_Q(s_{AQ}, s_{BQ}) = C_Q(s'_{AQ}, s'_{BQ})$ from (4), which contradicts the condition $C_I(s_{AI}, s_{BI}) \neq C_I(s'_{AI}, s'_{BI})$ or $C_Q(s_{AQ}, s_{BQ}) \neq C_Q(s'_{AQ}, s'_{BQ})$ in the minimum distance function (14).

Thus, employing the modulo operation for PNC, the minimum distance can avoid zero minimum distance, so that it will be greater than the second minimum distance, which can be easily found as $2g^2/\mathcal{E}\sigma_R^2$ $\left(d^2_{\min,\text{MA LD}} \geq 2g^2/\mathcal{E}\sigma_R^2\right)$ when $Z_I = 1$ and $Z_Q = 0$ in (14). Although we do not check whether the modulo operation can remove the second minimum distance or other candidates of the minimum distance, it is not necessary since it is enough for the proof to have the relation $d_{\min,\text{MA LD}} \geq 2g^2/\mathcal{E}\sigma_R^2$.

Next, we examine the value of the maximum upper bound of the minimum distance. Plugging $g = \sqrt{P_A^\dagger}\mathbf{g}_R^{\dagger H}\mathbf{H}_A\mathbf{u}_A^\dagger = \sqrt{P_B^\dagger}\mathbf{g}_R^{\dagger H}\mathbf{H}_B\mathbf{u}_B^\dagger$ into the upper bound in (10), the maximum value can be expressed as $g^2\delta^2_{\min}/\sigma_R^2$. Since we have $\delta^2_{\min} = 2/\mathcal{E}$ by definition and (1), it follows $U^\dagger_{\text{MA,LD}} = 2g^2/\mathcal{E}\sigma_R^2$ which is the same as the lower bound of the minimum distance. After all, we can conclude that the parameters obtained to maximize the upper bound of the minimum distance are optimal when combined with the modulo operation for PNC.

We can check this again in Fig. 3 which describes the inphase or quadrature of 16QAM. In this figure, some points are overlapped due to the same effective channel, which we refer to as co-points hereafter. The modulo operation design allocates the co-points to the same encoding output, so that errors caused by these points can be avoided. Also, all the distances b_i become the same maximized values. It is interesting to note that the obtained maximum value of the minimum distance is the minimum SNR in (10) with no MA interference. Thus, we can conclude that the modulo operation for PNC removes the MA interference completely, and provides the optimal solution which achieves the upper bound.

Now, we present detection for the LD system. For the LD system, the received signal is simply expressed as $y_R = g\Big(x_{AI} + x_{BI} + j(x_{AQ} + x_{BQ})\Big) + z_R$. Considering that $s_{Ak} + s_{Bk}$ is necessary for the modulo operation in (4), we can just detect $x_{Rk} \triangleq x_{Ak} + x_{Bk}$ as

$$\widehat{x}_{Rk} = \arg\max_{x_{Rk}} |y_{Rk} - g x_{Rk}|^2 \quad \text{for} \quad k = I \text{ and } Q, \tag{15}$$

where y_{RI} and y_{RQ} represent the real and imaginary parts of y_R, respectively. Then, $s_{Ak} + s_{Bk}$ can be obtained using $s_{Ak} + s_{Bk} = \sqrt{\mathcal{E}/2}\ \widehat{x}_{Rk} + \sqrt{M} - 1$. Since co-points are generated by the same effective channel as shown in Fig. 3, the search size of the LD system becomes $4\sqrt{M} - 2$.

C. Optimum closed-form solutions for the MA stage with LD systems

Although an iterative procedure is required to solve (11) in general, closed form solutions are available for two special cases, which will be described in this subsection. For these solutions, we will first transform our problem into a new one by adding one more constraint. Let us define a feasible set of $\mathbf{t}_R$ as $\mathbb{U} = \mathbb{W} \cup \overline{\mathbb{W}}$ where $\mathbb{W} \triangleq \{\mathbf{t}_R \in \mathbb{C}^{N_R \times 1} \mid \alpha \geq \beta,\ ||\mathbf{t}_R||^2 = 1\}$ and $\overline{\mathbb{W}} \triangleq \{\mathbf{t}_R \in \mathbb{C}^{N_R \times 1} \mid \alpha < \beta,\ ||\mathbf{t}_R||^2 = 1\}$. For $\mathbf{t}_R \in \mathbb{W}$, we denote the maximum value of $\min(\alpha, \beta)$ as O_β. Also, for $\mathbf{t}_R \in \overline{\mathbb{W}}$, O_α is defined as the maximum value. Then, the problem (11) can be expressed as $\mathbf{t}_R = \arg\max(O_\alpha, O_\beta)$. Before solving this, we check two possible solutions. The first possible solution is the eigenvector corresponding to the maximum eigenvalue of $\mathbf{U}_R\Sigma_B\Sigma_B^H\mathbf{U}_R^H$, denoted as $\mathbf{u}_{R1}$, which maximizes β [25]. If $\mathbf{u}_{R1}$ is an element of $\mathbb{W}$ ($\mathbf{u}_{R1} \in \mathbb{W}$), O_β is obtained with $\mathbf{t}_R = \mathbf{u}_{R1}$. Then, $\mathbf{u}_{R1}$ would be a solution of (11) since $O_\beta > O_\alpha$ in this case.

Next, we can check the second possible solution $\mathbf{e}_1$ where $\mathbf{e}_1 \triangleq [1\ 0\ \cdots\ 0]^T$ is the eigenvector corresponding to the maximum eigenvalue of $\Sigma_A\Sigma_A^H$, which maximizes α. Similar to the first possible solution, if $\mathbf{e}_1 \in \overline{\mathbb{W}}$, the optimal $\mathbf{t}_R$ becomes $\mathbf{e}_1$. If no solutions are available (i.e. $\mathbf{u}_{R1} \notin \mathbb{W}$ and $\mathbf{e}_1 \notin \overline{\mathbb{W}}$), we apply the following theorem.

Theorem 1: Consider a problem which maximizes $\min(\mathbf{t}_R^H\mathbf{K}_A\mathbf{t}_R,\ \mathbf{t}_R^H\mathbf{K}_B\mathbf{t}_R)$ for $\mathbf{t}_R \in \mathbb{U}$, where $\mathbf{K}_A \triangleq \Sigma_A\Sigma_A^H/\sigma_A^2$ and $\mathbf{K}_B \triangleq \mathbf{U}_R\Sigma_B\Sigma_B^H\mathbf{U}_R^H P_{BA}/\sigma_B^2$ are positive semi-definite (PSD). Let us denote $\mathbf{k}_i$ as the eigenvector corresponding to the maximum eigenvalue of $\mathbf{K}_i$ for $i = A$ and B. Then, for the case of $\mathbf{k}_A \notin \overline{\mathbb{W}}$ and $\mathbf{k}_B \notin \mathbb{W}$, a solution exists in the reduced set $\mathbb{V} \triangleq \{\mathbf{t}_R \in \mathbb{C}^{N_R \times 1} \mid \alpha = \beta,\ ||\mathbf{t}_R||^2 = 1\}$.

Proof: See Appendix. ■

Based on the result of Theorem 1, we obtain one more condition $\mathbf{t}_R^H\mathbf{K}_A\mathbf{t}_R = \mathbf{t}_R^H\mathbf{K}_B\mathbf{t}_R$. Then, if $\mathbf{u}_{R1} \notin \mathbb{W}$ and $\mathbf{e}_1 \notin \overline{\mathbb{W}}$, another solution of our problem (11) can be expressed as

$$\begin{aligned} \mathbf{t}_R^\dagger &= \arg\max_{\mathbf{t}_R} \mathbf{t}_R^H\mathbf{K}_A\mathbf{t}_R = \arg\max_{\mathbf{t}_R} \mathbf{t}_R^H\mathbf{K}_B\mathbf{t}_R \\ &\quad \text{s.t. } ||\mathbf{t}_R||^2 = 1 \text{ and } \mathbf{t}_R^H(\mathbf{K}_A - \mathbf{K}_B)\mathbf{t}_R = 0. \end{aligned} \tag{16}$$

Note that the original problem is transformed into a simple one (16). Then, the algorithm to find the optimal $\mathbf{t}_R^\dagger$, which we will refer to as the max-min algorithm is summarized below.

If $\mathbf{u}_{R1} \in \mathbb{W}$ or $\mathbf{e}_1 \in \overline{\mathbb{W}}$

$$\mathbf{t}_R^\dagger = \begin{cases} \mathbf{u}_{R1} & \text{for } \mathbf{u}_{R1} \in \mathbb{W} \\ \mathbf{e}_1 & \text{for } \mathbf{e}_1 \in \overline{\mathbb{W}} \end{cases}$$

else

$\mathbf{t}_R$ is obtained by solving the problem (16).

In the max-min algorithm, it can be easily shown that the two cases $\mathbf{u}_{R1} \in \mathbb{W}$ and $\mathbf{e}_1 \in \overline{\mathbb{W}}$ do not occur simultaneously. Once $\mathbf{t}_R$ is computed from this algorithm, we can also calculate $\mathbf{g}_R^\dagger$ and $\mathbf{u}_i^\dagger$.

Now, a closed form solution of (16) will be presented for $N_R = 2$ or $r_A = r_B = 1$, where r_i stands for the rank of $\mathbf{H}_i$ for $i = A$ and B. In (16), since $\mathbf{K}_A$ and $\mathbf{K}_B$ are Hermitian matrices, $\mathbf{K}_L \triangleq \mathbf{K}_A - \mathbf{K}_B$ is also Hermitian. Denoting eigenvalue decomposition of $\mathbf{K}_L$ as $\mathbf{K}_L = \mathbf{U}_L\mathbf{C}_L\mathbf{U}_L^H$, we set $\mathbf{t}_R = \mathbf{U}_L\overline{\mathbf{t}}_R$ for simple derivations, and then we will find $\overline{\mathbf{t}}_R$ to satisfy (16). With this relation, the problem (16) can be transformed into

$$\max_{\overline{\mathbf{t}}_R} \ \overline{\mathbf{t}}_R^H\mathbf{U}_L^H\mathbf{K}_A\mathbf{U}_L\overline{\mathbf{t}}_R \qquad \text{s.t. } ||\overline{\mathbf{t}}_R||^2 = 1 \text{ and } \sum_{k=1}^{N_R} c_k|t_k|^2 = 0 \tag{17}$$

where t_k is the k-th element of $\overline{\mathbf{t}}_R$ and c_k stands for the k-th eigenvalue of $\mathbf{K}_L$.

1) $N_R = 2$: First, we check existence of a solution. In this case, from the constraint of (17), it can be shown that the magnitudes of t_1 and t_2 are calculated as $|t_1| = \sqrt{\frac{c_2}{c_2-c_1}}$ and $|t_2| = \sqrt{\frac{c_1}{c_1-c_2}}$. If there do not exist solutions $\left(\text{i.e., } \frac{c_2}{c_2-c_1} < 0 \text{ or } \frac{c_1}{c_1-c_2} < 0\right)$, it can be checked that $\mathbf{t}_R^H\mathbf{K}_A\mathbf{t}_R$ is always smaller or always greater than $\mathbf{t}_R^H\mathbf{K}_B\mathbf{t}_R$ for all $\mathbf{t}_R$. These cases satisfy $\mathbf{u}_{R1} \in \mathbb{W}$ or $\mathbf{e}_1 \in \overline{\mathbb{W}}$ in Section III-B. Since the problem (16) has the conditions $\mathbf{u}_{R1} \notin \mathbb{W}$ and $\mathbf{e}_1 \notin \overline{\mathbb{W}}$, there exists a solution.

Next, we determine the phase of t_i for (17). Since $\mathbf{U}_L$ and $\mathbf{K}_A$ are 2×2 matrices, the problem can be simplified to

$$\max_{\overline{\mathbf{t}}_R} \overline{\mathbf{t}}_R^H\mathbf{U}_L^H\mathbf{K}_A\mathbf{U}_L\overline{\mathbf{t}}_R = \max_{w_1,w_2} \frac{\sigma_{A1}^2|w_1|^2 + \sigma_{A2}^2|w_2|^2}{\sigma_A^2} \tag{18}$$

where σ_{ik} equals the k-th singular value of $\mathbf{H}_i$ for $i = A$ and B, and w_j indicates the j-th element of $\mathbf{U}_L\overline{\mathbf{t}}_R$. Substituting $|w_1|^2 + |w_2|^2 = 1$ into (18), it follows that the problem is equivalent to maximizing $|w_1|^2 = |\mathbf{u}_{L1}^H\overline{\mathbf{t}}_R|^2$ with $\sigma_{A1}^2 \geq \sigma_{A2}^2 \geq 0$, where $\mathbf{u}_{L1}^H$ is defined as the first row of $\mathbf{U}_L$. Denoting u_i^* as the i-th element of $\mathbf{u}_{L1}$, we have $|w_1|^2 = |u_1t_1 + u_2t_2|^2$. Then, to maximize $|w_1|^2$, we choose the phase of t_i as $\angle t_i = -\angle u_i$. Finally, the optimal solution of $\overline{\mathbf{t}}_R$ for $N_R = 2$ can be calculated by

$$\overline{\mathbf{t}}_R = \left[\sqrt{\frac{c_2}{c_2-c_1}}\exp(-j\angle u_1) \quad \sqrt{\frac{c_1}{c_1-c_2}}\exp(-j\angle u_2)\right]^T.$$

2) $r_A = 1$ and $r_B = 1$: This case encompasses the system with $N_R = 1$ or $N_A = N_B = 1$. First, we will show that the rank of $\mathbf{K}_L$ equals two. From $r_A = r_B = 1$, $\mathbf{K}_L$ can be expressed as

$$\mathbf{K}_L = \begin{bmatrix}\mathbf{e}_1 & \mathbf{u}_{R1}\end{bmatrix} \begin{bmatrix} \frac{\sigma_{A1}^2}{\sigma_A^2} & 0 \\ 0 & -\frac{P_{BA}\sigma_{B1}^2}{\sigma_B^2}\end{bmatrix} \begin{bmatrix}\mathbf{e}_1^H \\ \mathbf{u}_{R1}^H\end{bmatrix}. \tag{19}$$

Since the channel is independently distributed, $\mathbf{e}_1$ and $\mathbf{u}_{R1}$ are linearly independent. Thus, the rank of $\mathbf{K}_L$ in (19) is equal to 2. Then, one constraint of (17) is given by $c_1|t_1|^2 + c_2|t_2|^2 = 0$ regardless of N_R.

Also, from $r_A = 1$, the cost function in (17) becomes $\sigma_{A1}^2\overline{\mathbf{t}}_R^H\mathbf{u}_{L1}\mathbf{u}_{L1}^H\overline{\mathbf{t}}_R/\sigma_A^2$. Similar to the case of $N_R = 2$, $\angle t_i$ is selected as $\angle t_i = -\angle u_i$. Substituting these phase values into $\mathbf{u}_{L1}^H\overline{\mathbf{t}}_R$, the problem for $|t_i|$ is formulated as

$$\begin{aligned}&\max_{\overline{\mathbf{t}}_R} \overline{\mathbf{t}}_R^H\mathbf{u}_{L1}\mathbf{u}_{L1}^H\overline{\mathbf{t}}_R \\ = \ &\max_{|t_i|} \sum_{i=1}^{N_R} |u_i||t_i| \\ &\text{s.t. } ||\overline{\mathbf{t}}_R||^2 = 1 \text{ and } c_1|t_1|^2 + c_2|t_2|^2 = 0.\end{aligned} \tag{20}$$

Using a Lagrangian method for this problem, $|t_i|$ can be calculated as

$$|t_1| = \frac{|u_1| + \sqrt{-\frac{c_1}{c_2}}|u_2|}{\lambda\left(1 - \frac{c_1}{c_2}\right)}, \quad |t_2| = \frac{\sqrt{-\frac{c_1}{c_2}}|u_1| - \frac{c_1}{c_2}|u_2|}{\lambda\left(1 - \frac{c_1}{c_2}\right)}$$

$$\text{and} \quad |t_i| = \frac{|u_i|}{\lambda} \quad \text{for} \quad i = 3, 4, \cdots, N_R$$

where

$$\lambda = \sqrt{\left(|u_1| + \sqrt{-\frac{c_1}{c_2}}|u_2|\right)^2 + \sum_{i=3}^{N_R}|u_i|^2}.$$

D. Suboptimal strategies for the MA stage with LD

In the previous subsection, closed-form solutions have been presented for two special cases. For more general configurations, however, the problem (11) or (16) still requires an iterative process. In this subsection, we propose a non-iterative suboptimal LD scheme which solves the problem (16) with reduced complexity. Basically, this suboptimal solution is assumed to be used with the max-min algorithm.

First, assuming that one link is turned off, we find a solution for the other link without interference. The optimal solution for the respective link can easily be obtained by the method in [25]. A solution for link A, which maximizes $\mathbf{K}_A$, is calculated as $\mathbf{t}_{R,A}^1 = \mathbf{e}_1$. Also, $\mathbf{t}_{R,B}^1 = \mathbf{u}_{R1}$ optimizes link B. To make the best of both links, two solutions can be combined as

$$\mathbf{t}_R^1 = \frac{\mathbf{t}_{R,A}^1 + \mathbf{t}_{R,B}^1}{||\mathbf{t}_{R,A}^1 + \mathbf{t}_{R,B}^1||}.$$

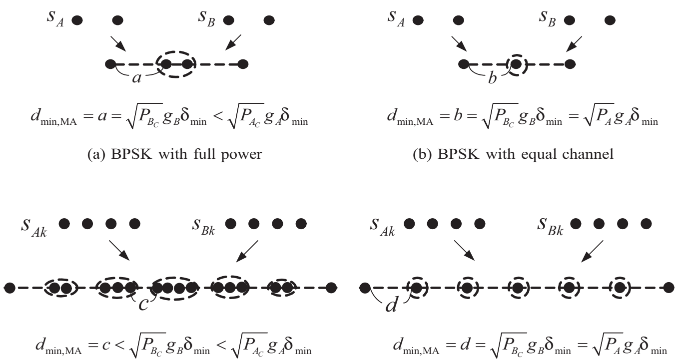
To further improve this combined solution, we consider the following bound of $\min(\alpha, \ \beta)$ in (11)

$$\min(\alpha, \beta) \leq \frac{\alpha + \beta}{2},$$

where the equality holds if and only if $\alpha = \beta$. By maximizing $\alpha + \beta$ instead of $U_{\text{MA,LD}}$, we have another solution, denoted as $\mathbf{t}_R^2$, which is the eigenvector corresponding to the maximum eigenvalue of $\mathbf{K}_A + \mathbf{K}_B$ [25]. Now, a new suboptimal scheme determines $\mathbf{t}_R$ by selecting one of the two solutions $\mathbf{t}_R^1$ and $\mathbf{t}_R^2$ as

$$\mathbf{t}_R^\dagger = \arg\max_{\mathbf{t}_R^i} U_{\text{MA,LD}}(\mathbf{t}_R^i).$$

Interestingly, it will be shown in the simulation section that the performance of this simple selection method is quite close to the optimal LD algorithm in Section III-B.



(a) BPSK with full power (b) BPSK with equal channel (c) 4PAM with full power (d) 4PAM with equal channel

Fig. 4. Minimum distance comparison between the full power and the equal channel case

E. Discussion on full power strategies for LD systems at the MA stage

In (12), we have already calculated the transmit power for the LD system, which maintains the maximum upper bound $U_{\mathrm{MA,LD}}$, and have shown that the minimum distance can achieve this maximum value by the modulo operation. However, it is not full power transmission. In this subsection, we will show that the full power strategy not only wastes power in terms of the minimum distance, but also degrades the performance for high modulation levels.

For simple explanations, we rewrite the equation (9) as

$$y_R = \sqrt{P_A} g_A x_A + \sqrt{P_B} g_B x_B + z_R,$$

where g_A and g_B represent the effective channel. Without loss of generality, we assume $\sqrt{P_{A_C}} g_A > \sqrt{P_{B_C}} g_B$ with full power for BPSK. The minimum distance is $a = \sqrt{P_{B_C}} g_B \delta_{\min}$ as shown in Fig. 4 (a). In this figure, the points inside the dashed circle indicate the clustering made by the modulo operation. Thus, no error is caused by these points. In Fig. 4 (b), we also depict the received constellation points with the equal channel case which results from the power allocation (12). Then, we can check that lower power P_A satisfying $\sqrt{P_A} g_A = \sqrt{P_{B_C}} g_B$ in Fig. 4 (b) has the same minimum distance as the full power case with P_{A_C}. Thus, we can see that the full power is not necessary in terms of the minimum distance for BPSK and QPSK.

Moreover, for high modulation levels, the full power transmission can degrade the performance compared to the equal channel case. In Figures 4 (c) and (d) which illustrate the inphase of 16QAM systems, the full power case exhibits smaller minimum distance than the equal channel case due to the spread of the points, which results in a performance loss. Thus, we conclude that it is important to adopt power control for higher modulation levels. This will be confirmed later in the simulation section.

F. Optimization of the BC Stage

In the BC stage, the RN transmits the processed signal $\sqrt{P_R}\mathbf{u}_R\mathbf{x}_R$ to the ENs simultaneously. Similar to the MA stage, we consider the squared distance between the constellation points at the ENs defined as

$$\begin{aligned} &d^2_{\mathrm{BC},i}(s_R, s'_R) \\ &= |\sqrt{P_R}\mathbf{g}_i^H \mathbf{H}_i^H \mathbf{u}_R(\delta(s_{RI}, s'_{RI}) + j\delta(s_{RQ}, s'_{RQ}))|^2/\sigma_i^2, \quad (21) \end{aligned}$$

for $i = A$ and B. Then, we define the following minimum distance which is the most dominant factor for the SER of the BC stage

$$d^2_{\min,\mathrm{BC}} \triangleq \min\left(d^2_{\min,\mathrm{BC},A},\ d^2_{\min,\mathrm{BC},B}\right) \quad (22)$$

where $d^2_{\min,\mathrm{BC},i} \triangleq \min d^2_{\mathrm{BC},i} = |\sqrt{P_R}\mathbf{g}_i^H \mathbf{H}_i^H \mathbf{u}_R \delta_{\min}|^2/\sigma_i^2$ for $i = A$ and B.

Now, we identify $\mathbf{g}_i$ and $\mathbf{u}_R$ which maximize $d^2_{\min,\mathrm{BC}}$. Similar to the MA stage of the LD system, we set $\mathbf{g}_i = \mathbf{V}_i \mathbf{t}_i$ and $\mathbf{u}_R = \mathbf{U}_A \mathbf{f}_R$. Substituting these expressions to (5), the received signal is obtained as

$$\begin{aligned} &y_A = \sqrt{P_R}\mathbf{t}_A^H \Sigma_A^H \mathbf{f}_R x_R + z_A \\ \text{and} \qquad &y_B = \sqrt{P_R}\mathbf{t}_B^H \Sigma_B^H \mathbf{U}_R^H \mathbf{f}_R x_R + z_B. \end{aligned}$$

Similar to Section III-B, to maximize $d^2_{\min,\mathrm{BC},A}$ and $d^2_{\min,\mathrm{BC},B}$, the received vectors are given as $\mathbf{t}_A = \frac{\Sigma_A^H \mathbf{f}_R}{\sqrt{\mathbf{f}_R^H \Sigma_A \Sigma_A^H \mathbf{f}_R}}$ and $\mathbf{t}_B = \frac{\Sigma_B^H \mathbf{U}_R^H \mathbf{f}_R}{\sqrt{\mathbf{f}_R^H \mathbf{U}_R \Sigma_B \Sigma_B^H \mathbf{U}_R^H \mathbf{f}_R}}$. Then, using these solutions, (22) can be expressed as

$$\begin{aligned} &d^2_{\min,\mathrm{BC}} \\ &= P_R \delta^2_{\min} \min\left(\frac{\mathbf{f}_R^H \Sigma_A \Sigma_A^H \mathbf{f}_R}{\sigma_A^2},\ \frac{\mathbf{f}_R^H \mathbf{U}_R \Sigma_B \Sigma_B^H \mathbf{U}_R^H \mathbf{f}_R}{\sigma_B^2}\right). \quad (23) \end{aligned}$$

Interestingly, maximizing $d^2_{\min,\mathrm{BC}}$ is very similar to the procedure of $\max\min(\alpha, \beta)$ in Section III-B. Thus, the transmit filter $\mathbf{f}_R$ can be solved by the results in Section III-B, III-C and III-D. In this case, the relay transmits with full power ($P_R = P_{R_C}$).

IV. Simulation Results

In this section, we evaluate the following systems in terms of the BER: The MLD system, the LD system with an iterative method (opt-LD), and the suboptimum LD system (sub-LD). The individual power constraints are applied as $P_{A_C} = P_{B_C} = P_{R_C} = P$. Also, the average SNR is defined as P/σ^2 with $\sigma_A^2 = \sigma_B^2 = \sigma_R^2 = \sigma^2$. We use a notation of $N_A \times N_R \times N_B$ for representing antenna configurations in this section.

In Fig. 5, we plot the results of the proposed systems with BPSK for $3 \times 3 \times 3$ relaying networks. First, we can check in this figure that MLD exhibits the best performance as expected. It is interesting to note that the performance of the opt-LD is the same as that of MLD in this configuration. From this result, we may check that our approach of mapping into one complex dimension by LD and removing the MA interference by PNC is effective. Also, it can be checked that the performance of the sub-LD is within 1 dB at a BER of 10^{-4} compared to the opt-LD with much lower complexity. In addition, the performance of two-way AF systems with the optimal precoding in [9] for one stream is presented, which require an iterative gradient method. We can see that our

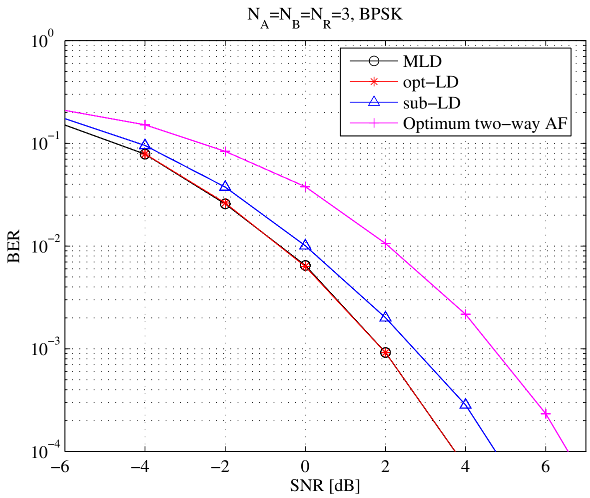


Fig. 5. BER performance of the $3 \times 3 \times 3$ two-way relaying networks with BPSK

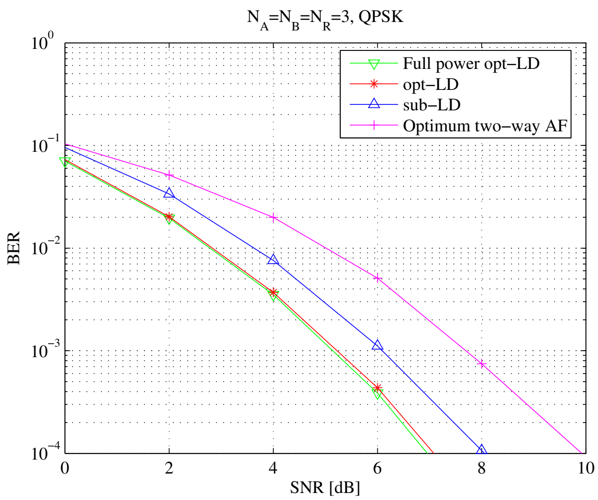


Fig. 6. BER performance of the $3 \times 3 \times 3$ two-way relaying networks with QPSK

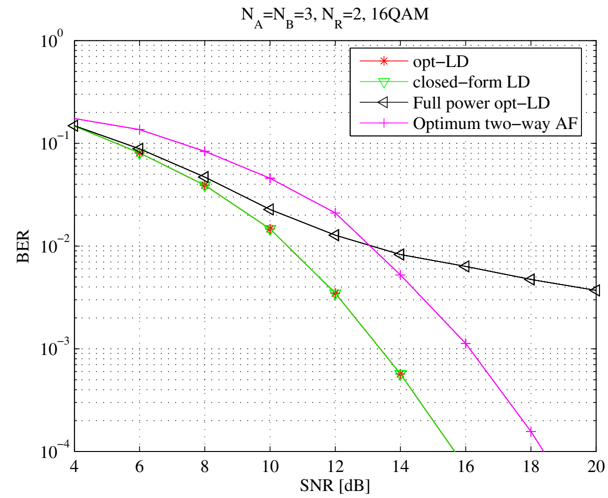


Fig. 7. BER performance of the proposed $3 \times 2 \times 3$ two-way relaying networks with 16QAM

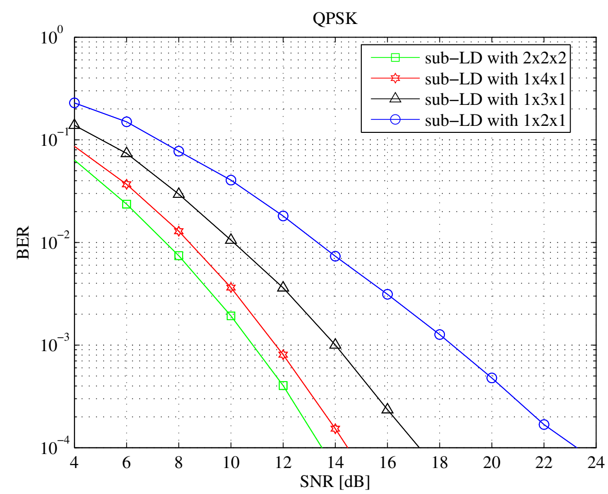


Fig. 8. Diversity comparison of the proposed suboptimal systems with QPSK

proposed systems provide 2-3 dB gains at a BER of 10^{-4} over conventional two-way AF systems. This performance gain can be attributed to the fact that the RN of two-way AF adds two symbols from the ENs as one transmitting signal, while the RN with PNC encodes one symbol which consumes less power than the two-way AF case. Also, since the AF systems are not able to remove the noise at the RN, its power is boosted when the RN transmits the signal.

Fig. 6 depicts the BER performance of the proposed systems with QPSK for $3 \times 3 \times 3$ antenna configurations. Note that MLD solutions are not available for QPSK. Similar to Fig. 5, the proposed systems achieve 2-3 dB gains at a BER of 10^{-4} over two-way AF systems. As expected from the analysis in Section III-E, full power transmission has almost the same performance as the opt-LD with the equal effective channel. The slight gap between two systems reflects that the BER may not be exactly the same, even if these systems have the same minimum distance.

In Fig. 7, the BERs are exhibited for $3 \times 2 \times 3$ systems. In this figure, the LD systems with a closed-form solution is referred to as the closed-form LD. It can be shown that the closed-form LD has the same performance as the optimal approach, which confirms that our derived solution is correct. Also, we can check in this figure that the performance of full power transmission is severely degraded at high SNR due to the MA interference, as discussed in Section III-E. Thus, it is important to adopt power control for high modulation levels. In addition, we can see that the proposed systems have a 3 dB gain compared to the optimal two-way AF systems.

Finally, Fig. 8 evaluates the sub-LD with various antenna configurations. It can be noted in the plot that although suboptimal solutions are adopted, our systems achieve a full diversity order $N_R \min(N_A, N_B)$ as shown in [26]. From the simulation results, it is clear that our proposed systems are very effective for relaying networks.

V. Conclusion

In this paper, we have considered a design of two way wireless relaying systems. Assuming the modulo operation as PNC, we have first proposed iterative methods for the optimal precoding which maximizes an upper bound in individual power constraints. We separately optimize the filters for the MA and BC stages. Then, we have proved that the precoding of the MA stage is jointly optimal with the modulo operation method for PNC. Also, for the cases of $N_R = 2$ or $r_A = r_B = 1$, we have derived a closed-form solution for the precoding. To deal with exact solutions, we change our max min problem into a simple maximizing problem by imposing additional constraints. Also, to lower the complexity, we have proposed a simple suboptimal precoding whose performance is within 1 dB compared to the optimal systems. It has been shown from the simulation results that the proposed systems achieve 2-3 dB gains at a BER of 10^{-4} over the optimal AF systems. Also, we have shown that a full power strategy in the system with individual power constraint degrades the performance. An extension to the case of non-equal modulation levels is left for further study.

Appendix
Proof of Theorem 1

First, we consider $\mathbf{t}_R \in \mathbb{W}$ which maximizes $\mathbf{t}_R^H \mathbf{K}_B \mathbf{t}_R$. Although $\mathbf{t}_R = \mathbf{k}_B$ maximizes $\mathbf{t}_R^H \mathbf{K}_B \mathbf{t}_R$, this cannot be a solution because of the assumption $\mathbf{k}_B \notin \mathbb{W}$. Let us denote

$$\mathbf{k}_Z = \frac{te^{j\theta}\mathbf{k}_B + (1-t)\mathbf{k}_W}{|te^{j\theta}\mathbf{k}_B + (1-t)\mathbf{k}_W|}$$

for $\mathbf{k}_W \in \mathbb{W}$, $-\pi \leq \theta \leq \pi$ and $0 \leq t \leq 1$. Then, defining $M \triangleq \mathbf{k}_B^H \mathbf{K}_B \mathbf{k}_B$ and $l \triangleq \mathbf{k}_W^H \mathbf{k}_B$, we have $\mathbf{k}_W^H \mathbf{K}_B \mathbf{k}_B = Ml$. From these, $\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$ can be expressed as

$$\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z = \frac{Mt^2 + m(1-t)^2 + 2\Re[l\exp(j\theta)]t(1-t)}{(2-2\Re[l\exp(j\theta)])t^2 - (2-2\Re[l\exp(j\theta)])t + 1}$$

where $m \triangleq \mathbf{k}_W^H \mathbf{K}_B \mathbf{k}_W$, and $\Re(\cdot)$ stands for a real element.

After some mathematical manipulations, the derivative of $\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$ with respect to t is calculated by

$$\frac{d}{dt}\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z = \frac{-2(M-m)\{(1-\Re[le^{j\theta}])t + \Re[le^{j\theta}]\}(t-1)}{\{(2-2\Re[le^{j\theta}])t^2 - (2-2\Re[le^{j\theta}])t + 1\}^2}. \quad (24)$$

In this equation, since the maximum of $\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$ is M, and $\mathbf{K}_B$ is PSD, it follows $M \geq m \geq 0$. By using this inequality, (24) is positive for $\frac{-\Re[l\exp(j\theta)]}{1-\Re[l\exp(j\theta)]} \leq t \leq 1$. Then, it can be shown that for any $\theta \in \Omega \triangleq \{\theta | 0 \leq \Re[l\exp(j\theta)] \leq 1\}$, $\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$ is a monotonically increasing function of t for $0 \leq t \leq 1$. This implies that $\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$ is larger than $\mathbf{k}_W^H \mathbf{K}_B \mathbf{k}_W$ for an arbitrary $\mathbf{k}_W \in \mathbb{W}$, since $\mathbf{k}_Z$ is equal to $\mathbf{k}_W$ for $t = 0$.

Next, for an arbitrary $\mathbf{k}_W \in \mathbb{W}$, we check if there exists $\mathbf{k}_Z$ in the reduced set $\mathbb{V}$ for $0 \leq t \leq 1$ and $\theta \in \Omega$. Denoting n_1, n_2 and n_3 as $n_1 \triangleq \mathbf{k}_W^H(\mathbf{K}_A - \mathbf{K}_B)\mathbf{k}_W$, $n_2 \triangleq -\mathbf{k}_B^H(\mathbf{K}_A - \mathbf{K}_B)\mathbf{k}_B$ and $n_3 \triangleq \Re[\mathbf{k}_W^H(\mathbf{K}_A - \mathbf{K}_B)\mathbf{k}_B]$, respectively, the equation $\mathbf{k}_Z^H(\mathbf{K}_A - \mathbf{K}_B)\mathbf{k}_Z = 0$ can be expressed as

$$(n_1 - n_2 - 2n_3)t^2 - 2(n_1 - n_3)t + n_1 = 0. \quad (25)$$

One solution for this equation is $t^\dagger = 1/(1 + \sqrt{(\frac{n_3}{n_1})^2 + n_2} - \frac{n_3}{n_1})$. Since we have $\mathbf{k}_W \in \mathbb{W}$ and $\mathbf{k}_B \in \overline{\mathbb{W}}$, both n_1 and n_2 are non-negative. Then, it follows $0 \leq t^\dagger \leq 1$. It can be noticed that $t^\dagger$ is valid for all $\theta \in \Omega$. Thus, there exists $\mathbf{k}_Z$ which satisfies $\mathbf{k}_Z^H \mathbf{K}_A \mathbf{k}_Z = \mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$.

Finally, from the above derivations, for an arbitrary $\mathbf{k}_W \in \mathbb{W}$, there exists $\mathbf{k}_Z$ in the reduced feasible set $\mathbf{k}_Z \in \mathbb{V}$ (i.e. $\mathbf{k}_Z^H \mathbf{K}_A \mathbf{k}_Z = \mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$), which satisfies that $\mathbf{k}_Z^H \mathbf{K}_B \mathbf{k}_Z$ is larger than $\mathbf{k}_W^H \mathbf{K}_B \mathbf{k}_W$. Thus, for $\mathbf{t}_R = \mathbf{k}_W \in \mathbb{W}$, to maximize the problem, a solution exists in the reduced set $\mathbb{V}$. For $\mathbf{t}_R \in \overline{\mathbb{W}}$, we can prove in a similar way.

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