Performance Analysis of Multiuser MIMO Systems with Zero Forcing Receivers

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Abstract—In this paper, we consider multiuser multi-input/multi-output antenna systems with zero-forcing receivers in downlink. In this case, to exploit multiuser diversity, spatial-division multiple access (SDMA) system allows to assign different users to a part of transmit antennas at the base station whereas spatial-division multiplexing (SDM) system assigns all antennas to single user’s data stream. In this paper, we present analytical frameworks to evaluate performance of these systems. We first analyze the performance of these two systems by deriving closed-form expressions of achievable throughput. Numerical results show that the derived expressions are very tight. In addition, we approximate the capacity expression of SDM and SDMA systems and compare the SDM with the optimal case.

I. INTRODUCTION

Most wireless communication systems have been designed to support high-speed packet data transmissions on a downlink channel. Those high-speed packet services can be provided by employing efficient usages of the transmission bandwidth. In such environments, much attention has been paid to multi-input/multi-output (MIMO) antenna systems [1].

For single user communications, it has been shown in [2] that the MIMO channels exhibit significant capacity gains over single antenna systems as the number of antennas increases. When channel state information (CSI) is known at the transmitter, the mutual information can be maximized by adapting the transmit power to the channel via the water-filling technique after decomposing the MIMO channel into multiple parallel independent eigenmodes via singular value decompositions (SVD) [1].

In [3], the authors approximate the capacity expression of MIMO channels in multiuser scenarios by observing that the distribution of the mutual information in a Rayleigh fading channel obeys the Gaussian distribution. According to the result in [3], the most efficient architecture for the multiuser MIMO system is the system with an equal number of antennas at both the transmitter and receiver. Therefore, in this paper, we consider a system where the same number of antennas are equipped at both the transmitter and receiver.

In contrast to the optimal MIMO system with the eigen beamforming operation, we employ zero-forcing (ZF) receivers for MIMO systems to provide simpler transmit/receive operations. Instead of performing the SVD operation, the ZF equalizer is utilized to decompose the MIMO channel into multiple parallel streams. With the ZF receiver, we can consider two system models for multiuser scenarios. Those are the spatial-division multiple access (SDMA) system which allows to assign different users to a part of transmit antennas at the base station (BS) and the spatial-division multiplexing (SDM) system which assigns all antennas to single user’s data stream. We analyze the performance of these two systems by comparing achievable throughput. To this end, we will derive closed form expressions for the achievable throughput of MIMO systems with multiple users and compare the performance among different system architectures. In addition, we approximate the throughput expressions of SDM and SDMA systems using the order statistics theory [4] to provide an insight into primary parameters which affect the system performance.

The numerical results show that the derived expressions are very tight compared with the results obtained by simulations and SDMA systems outperform SDM systems by around 3dB with four transmit antennas and 40 users. Also the achievable throughput approximation for the SDM describes well the real performance and it is shown that the performance comes within 2dB compared with the system employing the SVD as the number of users increases.

The paper is organized as follows: In section II, the system model for the MIMO is described. The performance analysis of MIMO systems with the ZF receiver is presented in section III. Finally, the numerical results and conclusion are presented in sections IV and V, respectively.
II. SYSTEM MODEL

Fig. 1 illustrates the MIMO communication systems with the ZF receiver comprising $N_t$ transmit and $N_r$ receive antennas. Denoting $\mathbf{H}$ as an $N_r \times N_t$ MIMO channel matrix whose $(i,j)$-th entry represents the channel response of the link between the $j$th transmit antenna and the $i$th receive antenna, the received signal model can be simply described as $\mathbf{y} = \mathbf{Hx} + \mathbf{n}$ where $\mathbf{n}$ denotes an independent and identically distributed (i.i.d) complex additive white Gaussian noise (AWGN) vector of length $N_r$ with $E[\mathbf{n}^H\mathbf{n}] = \sigma_n^2 \mathbf{I}_{N_r}$ and $\mathbf{x}$ stands for the transmitted symbol vector of length $N_t$ with $E[\mathbf{x}^H\mathbf{x}] = \sigma_x^2 \mathbf{I}_{N_t}$. Here $\mathbf{I}_n$ represents an identity matrix of size $n \times n$, and $(\cdot)^*$ denotes the Hermitian transpose of the matrix. Also we assume that the user data $\mathbf{x}$ and the noise $\mathbf{n}$ are uncorrelated with each other.

With the ZF equalizer $\mathbf{W}_{ZF}$, the output of the ZF receiver can be written as $\mathbf{z} = \mathbf{W}_{ZF}\mathbf{y} = \mathbf{x} + \mathbf{W}_{ZF}\mathbf{n}$ where $\mathbf{W}_{ZF}$ is obtained from the pseudo-inverse of $\mathbf{H}$ defined as $\mathbf{W}_{ZF} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$. Then the output signal-to-noise ratio (SNR) at the $i$th stream after the ZF processing can be expressed by [5]

$$\theta_i = \frac{\rho}{N_t[(\mathbf{H}^H\mathbf{H})^{-1}]_{ii}}$$

where $[\mathbf{A}]_{ij}$ denotes the $(i,j)$-th entry of $\mathbf{A}$ and $\rho$ represents the average received SNR. Assuming $N_r \geq N_t$, $\theta_i$ is a Chi-squared random variable with a degree of freedom equal to $2(N_r - N_t + 1)$ [5]. Here all $\theta_i$’s are i.i.d. Thus the probability density function (PDF) of $\theta_i$ can be written as

$$f_{\theta_i}(\theta) = \frac{N_t e^{\frac{\rho}{\rho(N_r - N_t)}}}{\rho(N_r - N_t)!} \left(\frac{\rho}{\rho}\right)^{N_r - N_t} e^{-\frac{\rho}{\rho}\theta}, \quad \theta > 0. \quad (1)$$

Since the MIMO channel is decomposed into $N_t$ parallel channels with the ZF equalizer, the ergodic capacity can be obtained as the sum capacity of $N_t$ parallel channels [5]

$$C_{ZF} = \sum_{n=1}^{N_t} E[\log_2(1 + \theta_n)] \text{ (bps/Hz)} \quad (2)$$

When the ZF receiver model in Fig. 1 is extended to the multiuser case, we consider the following two architectures:

- **SDM**: Data streams of one user are transmitted through all transmit antennas at the BS. With the Max-Rate scheduling criterion [3], the BS can select the user with the largest $\sum_n \log_2(1 + \theta_n)$ at each time slot.
- **SDMA**: The BS can select up to $N_t$ different user streams at a given time which can best exploit each channel. Denoting $\theta_n^k$ as the output SNR at the $n$th spatial subchannel of the $k$th user, the BS allocates the $n$th spatial channel to the user with the highest $\log_2(1 + \theta_n^k)$ for $n = 1, \cdots, N_t$.

III. PERFORMANCE ANALYSIS OF MULTIUSER MIMO SYSTEMS WITH LINEAR RECEIVERS

In this section, we analyze the performance of multiuser MIMO systems with ZF receivers. As mentioned earlier, an antenna configuration which exploits multiuser diversity most efficiently is the system employing an equal number of antennas at both the transmitter and receiver [3]. Therefore, from now on, we assume $N_t = N_r = N$ and all users experience statistically independent identical fading process. Applying the ZF equalizers at the receiver, $N$ spatial channels are generated. We refer to the data stream transmitted via the $n$th spatial channel as the $n$th stream.

A. Closed Form Expression for Multiuser Capacity

In the case of $N_t = N_r = N$, the PDF of the output SNR $\theta_n$ in (1) reduces to an i.i.d. exponentially distribution which can be expressed as

$$f_{\theta_n}(\theta) = \frac{N}{\rho} e^{-\theta N / \rho} \quad (3)$$

For the SDMA where the BS can select up to $N$ users to exploit each antenna, $N$ independent links with $K$ users whose output SNR for each user can be modeled as (3). Defining the $k$th order statistics $\theta_{n,K}$ as the $k$th smallest of $\theta_1, \theta_2, \cdots, \theta_n$ [4], $\theta_{n,K}$ represents the largest output SNR for the $n$th stream among $K$ users which is selected through the max-rate scheduler [3]. The max-rate scheduling scheme is an opportunistic scheduling scheme which maximizes the total throughput in the system.

According to the order statistics [4], the PDF of $\theta_{n,K}$ can be obtained as

$$f_{\theta_{n,K}}(\theta) = \frac{dF_{\theta_n}(\theta)}{d\theta} = KF_{\theta_n}(\theta)^{K-1} f_{\theta_n}(\theta) \quad (4)$$

By substituting (3) to (4), the capacity of the SDMA with $K$ users is expressed as

$$C_{SDMA}^{ZF}(K) = \frac{N}{\rho} \int_{0}^{\infty} \log_2(1 + \theta) e^{-\theta N / \rho} (1 - e^{-\theta N / \rho})^{K-1} d\theta$$

$$= \frac{K N^2}{\rho} \int_{0}^{\infty} \log_2(1 + \theta) e^{-\theta N / \rho} (1 - e^{-\theta N / \rho})^{K-1} d\theta \quad (5)$$

Using the binomial power series expansion and the integral equality [6, pp. 568]

$$\int_{0}^{\infty} e^{-\mu x} \ln(1 + \beta x) dx = \frac{1}{\mu} e^{\mu/\beta} E_1 \left(\frac{\mu}{\beta}\right)$$

where the exponential integral function $E_1(x)$ is defined as $\int_{x}^{\infty} e^{-t} t^{-1} dt$, (5) can be rewritten as

$$C_{SDMA}^{ZF}(K) = \frac{K N^2}{\rho} \log_2(e)$$

$$\times \int_{0}^{\infty} \ln(1 + \theta) e^{-\theta / \rho} \sum_{k=0}^{K-1} \binom{K-1}{k} (-1)^k e^{-\frac{\theta}{\rho} k} d\theta$$

$$= K N \log_2(e)$$

$$\times \sum_{k=0}^{K-1} (-1)^k \frac{1}{k+1} \binom{K-1}{k} \exp \left(\frac{N}{\rho}(k+1)\right)$$

$$\times E_1 \left(\frac{N}{\rho}(k+1)\right) \quad (6)$$
Note that $E_1(x)$ can be efficiently computed by the approximation as in [7, pp. 231].

In contrast, for SDM systems, the BS selects the user with the highest sum rate, $\sum_n \log_2(1 + \theta_n^n)$. However, it is a problem of great difficulty to describe an exact distribution of $\sum_n \log_2(1 + \theta_n^n)$. To simplify the problem, we modify the SDM scheduling scheme as follows:

- The BS selects the $k$th user with the largest $\sum_n \theta_n^n$ instead of $\sum_n \log(1 + \theta_n^n)$.
- Transmit power is distributed equally to each stream in order to make the received SNR equal for each stream.

Note that the right-hand side of (8) is the characteristic information as in [7, pp. 231].

Define a new random variable $\Theta_k$ which stands for a sum of all $\theta_n^n$'s as $\Theta_k = \sum_{n=1}^{N} \theta_n^n$. Based on the above assumptions, the capacity formula of the $K$-user SDM can be rewritten as

$$C_{ZF}^{SDM}(K) = E \left[ \max_k \left( \sum_{n=1}^{N} \log(1 + \theta_n^n) \right) \right]$$

$$\approx N \int_0^{\infty} \log_2 \left( 1 + \frac{x}{N} \right) f_{\Theta_K}(x) dx$$

where $f_{\Theta_K}(x)$ indicates the Kth order statistics of $\Theta_k$. Since all $\theta_n^n$'s are i.i.d, the characteristic function of $\Theta_k$ can be obtained by multiplying the characteristic functions of $\theta_n^n$'s. Therefore, the characteristic function $\psi_{\Theta_k}(\omega)$ is written as

$$\psi_{\Theta_k}(\omega) = \prod_{n=1}^{N} \psi_{\theta_n^n}(\omega) = (1 + \frac{\rho}{N} \omega)^{-N}. \tag{8}$$

Note that the right-hand side of (8) is the characteristic function of the gamma distribution [9] with the PDF

$$f_{\Theta_k}(\theta) = \left( \frac{N}{\rho} \right)^N \theta^{N-1} e^{-\theta/\rho} \frac{\Gamma(N)}{\Gamma(N)}$$

where $\Gamma(N) = (N-1)!$ for integer $N$ [7].

Since the cumulative density function (CDF) of (9) is given as [9]

$$F_{\Theta_k}(x) = \frac{\gamma(N, (N/\rho)x)}{\Gamma(N)}$$

where $\gamma(a, x)$ denotes an incomplete gamma function defined as $\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt$ [6], the PDF of the $K$th order statistics of $\Theta_k$ can be represented as

$$f_{\Theta_K, K}(x) = K \left( \frac{\gamma(N, (N/\rho)x)}{\Gamma(N)} \right)^{K-1} \left( \frac{N}{\rho} x^{N-1} e^{-N x/\rho} \right) \frac{\Gamma(N)}{\Gamma(N)} \prod_{i=0}^{K-1} \left( \frac{N}{\rho} x^{N-1} e^{-N x/\rho} \right)$$

To integrate (7) efficiently, the PDF in (10) is converted into [10]

$$f_{\Theta_K, K}(x) = K \frac{\Gamma(N)}{\Gamma(N)} \sum_{k=0}^{K-1} \sum_{i=0}^{N-1} (-1)^k \binom{K-1}{k} a_i^k \left( \frac{N}{\rho} \right)^{N+i} x^{N+i-1}$$

where $a_i^k$ for $0 \leq i \leq k(N-1)$ is recursively defined as

$$a_0^k = 1, \quad a_1^k = k,$$

$$a_i^k = \frac{1}{i} \sum_{n=1}^{\min(i, N-1)} \frac{n(k+1) - i}{n!} \frac{a_i^{n-i}}{\rho^n}, \quad 2 \leq i < k(N-1)$$

$$a_i^k = \frac{1}{\Gamma(N)^k}, \quad i = k(N-1). \tag{12}$$

By applying (11) to (7) and using the integral identity [11]

$$\int_0^{\infty} \ln(1 + t) e^{-\mu t} t^{a-1} dt = (n-1)! e^\mu \sum_{j=1}^{n} \frac{\Gamma(j-n, \mu)}{\mu^j}$$

where $\Gamma(\alpha, x)$ represents another incomplete gamma function which is defined as $\Gamma(\alpha, x) = \int_x^{\infty} t^{\alpha-1} e^{-t} dt$ [6], the capacity of the $K$-user SDM system is finally computed by

$$C_{ZF}^{SDM}(K) = \frac{K N \log_2(e)}{(N-1)!} \times \sum_{k=0}^{K-1} \sum_{i=0}^{N-1} (-1)^k \binom{K-1}{k} a_i^k \left( \frac{N^2}{\rho} \right)^{N+i}$$

$$\times \int_0^{\infty} \ln(1 + t) e^{-\mu t} t^{N+i-1} dt$$

$$\times \left( \frac{K N \log_2(e)}{(N-1)!} \times \sum_{k=0}^{K-1} \sum_{i=0}^{N-1} (-1)^k \binom{K-1}{k} a_i^k \left( \frac{N^2}{\rho} \right)^{N+i} \right)^{N+i-1}$$

$$\times e^{(k+1)N^2/\rho} \sum_{j=1}^{N+i} \frac{\Gamma(j-N-i, (k+1)N^2/\rho)}{\left( (k+1)N^2/\rho \right)^j} \tag{13}$$

### B. Capacity Bounds

Although the derived formulas for SDM and SDMA systems in (6) and (13) in the previous section are quite accurate, they do not provide much information about primary factors which affect the system performance. In this subsection, in order to gain an insight on multiuser diversity gains, we develop upper bounds for channel capacity for both the SDM and SDMA schemes with $K$ users.

Denoting the mean and variance of the capacity of single user systems as $\mu$ and $\sigma^2$, respectively, the system throughput with $K$ users is bounded as [4]

$$C^K \leq \mu + \frac{(K-1)\sigma}{\sqrt{2K-1}}. \tag{14}$$

For the special case, if the capacity of each user obeys a Gaussian distribution, a tighter bound can be obtained from the weak law of large numbers [3] as

$$C^K \approx \mu + \sqrt{2\sigma^2 \ln K}. \tag{15}$$

As the Shannon capacity can be approximated by $\log_2 \theta$ at high SNR ranges ($N <<< \rho$) [2], the average value of
the achievable throughput of the nth stream for the SDM is rewritten as
\[
\mu^2_{SDMA} = \int_0^\infty \log_2(1 + \theta) \left( \frac{N}{\rho} \right) \exp(-\frac{N}{\rho} \theta) d\theta \\
\simeq \log_2 e \int_0^\infty \ln \left( \frac{N}{\rho} \right) \exp(-\frac{N}{\rho} \theta) d\theta .
\]
Then, the mean of \( C^Z_{SDMA}(K) \) in (5) with \( N \) transmit antennas and \( K = 1 \) can be expressed as
\[
\mu^2_Z = N \times \mu^2_{SDMA}.
\]
Define \( \mu_{SDMA} \) and \( \sigma^2_{SDMA} \) as the mean and variance of \( C^Z_{SDMA}(1) \), respectively. Using the integral identity [6, pp.567]
\[
\int_0^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} (\zeta + \ln \mu)
\]
where \( \zeta = 0.5772 \cdots \) denotes the Euler constant, \( \mu_{SDMA} \) can be obtained as
\[
\mu_{SDMA} = N \log_2 e \left( \ln \frac{\rho}{N} + \zeta \right) .
\]
Also \( \sigma^2_{SDMA} \) can be computed as
\[
\sigma^2_{SDMA} = E[(N \log_2(1 + \theta))^2] - \mu^2_{SDMA} \\
= N^2 \left( \log_2 e \right)^2 \left( \frac{\pi^2}{6} + (\zeta + \ln N) \right)^2 - (\mu^2_{SDMA})^2 \\
= N^2 (\log_2 e)^2 \frac{\pi^2}{6} .
\]
where we have used the integral identity [6, pp.567]
\[
\int_0^\infty e^{-\mu x} (\ln x)^2 dx = \frac{1}{\mu} \left( \frac{\pi^2}{6} + (\zeta + \ln \mu)^2 \right) .
\]
In contrast, for SDM systems, the average capacity is approximated as
\[
\mu_{SDM} = N \int_0^\infty \log \left( 1 + \frac{\theta}{N} \right) \left( \frac{N}{\rho} \right) N \theta^{N-1} e^{-\theta N/\rho} (N-1)! d\theta \\
\simeq (\log_2 e) \left( \frac{N}{\rho} \right) N \frac{N}{(N-1)!} \int_0^\infty \ln \frac{\theta}{N} \theta^{N-1} e^{-\theta N/\rho} d\theta
\]
for \( N \ll \rho \). Using the Euler psi function \( \psi(z) \) in [7] and the integral identity [6, pp.569]
\[
\int_0^\infty x^{\nu-1} e^{-\mu x} \ln x dx = \mu^{-\nu} \Gamma(\nu) [\psi(\nu) - \ln \mu] ,
\]
(17) can be computed as
\[
\mu_{SDM} \simeq \log_2 e \left( \frac{N^2}{\rho} \right)^N \\
\times \frac{N}{(N-1)!} \Gamma(N) \left( \frac{N^2}{\rho} \right)^{-N} \left[ \psi(N) + \ln \frac{\rho}{N^2} \right] \\
= N \log_2 e \left[ \psi(N) + \ln \frac{\rho}{N^2} \right] .
\]
In addition, the variance \( \sigma^2_{SDM} \) is obtained as
\[
\sigma^2_{SDM} \simeq N^2 \left( E \left( \frac{\theta}{N} \right)^2 - \mu^2_{SDM} \right)
\]
where \( E \left( \frac{\theta}{N} \right)^2 \) is computed as
\[
E \left( \frac{\theta}{N} \right)^2 \simeq \frac{N}{(N-1)!} (\log_2 e)^2 \int_0^\infty \frac{\theta^{N-1} e^{-\theta N/\rho} d\theta}{(N-1)!} \\
= \frac{(\log_2 e)^2}{(N-1)!} \int_0^\infty (\ln x)^2 N^{N-1} e^{-N^2 x} dx .
\]
Applying the integral identity [6, pp.572]
\[
\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln x)^2 dx = \mu^{-\nu} \Gamma(\nu) \left( [\psi(\nu) - \ln \mu]^2 + \zeta(2, \nu) \right)
\]
where \( \zeta(z, q) \) represents the Riemann zeta function[7], (20) can be rewritten as
\[
E \left( \frac{\theta}{N} \right)^2 \simeq (\log_2 e)^2 \left( [\psi(N) - \ln \frac{N^2}{\rho}]^2 + \zeta(2, N) \right) .
\]
After some manipulations, substituting (21) to (19) yields
\[
\sigma^2_{SDM} \simeq N^2 (\log_2 e)^2 \left( \frac{\pi^2}{6} - \sum_{n=1}^{N-1} \frac{1}{n^2} \right)
\]
where we have used \( \sum_{n=1}^\infty 1/n^2 = \pi^2/6 \). As will be shown in section IV, the channel capacity of the SDM with \( K \) users can be closely expressed by the Gaussian approximation in (15) with (18) and (22).

The performance of the SDM and the SDMA can be compared by the following lemma.

Lemma 1 (Capacity Gain of the SDMA Systems): Define the capacity gain of the SDMA compared with the SDM as \( \eta \). Then we have
\[
\lim_{K \to \infty} \eta = \lim_{K \to \infty} \frac{C^K_{SDMA}}{C^K_{SDM}} > 1 .
\]
Proof: By substituting the results of (16) and (22) to (14), we have
\[
\lim_{K \to \infty} \frac{C^K_{SDMA}}{C^K_{SDM}} = \frac{\sigma_{SDMA}}{\sigma_{SDM}} \\
= \sqrt{\frac{\pi^2}{\pi^2 - 6 \sum_{n=1}^{N-1} (1/n^2) > 1} .
\]
Thus we can see that the capacity gain of SDMA systems over SDM systems increases as the number of antennas \( N \) grows.

IV. NUMERICAL RESULTS

In this section, we present numerical results to verify the performance analysis of MIMO systems with the linear equalizer at the receiver. To evaluate performance, we assume i.i.d Rayleigh fading channels which is supposed to be flat over transmission bandwith. Fig. 2 exhibits the capacity of
MIMO systems with four transmit and four receive antennas and 40 users. To verify the accuracy of the derived closed form expressions, we compare the analysis results with the Monte Carlo simulations. As shown in the figure, the derived formulas match well with two MIMO system models. Also we see that the performance gain of the SDMA system over the SDM is around 3dB.

In Fig. 3, the performance of SDM systems with various configurations is illustrated. Fig. 3 shows that approximations with (15) describe well the SDM systems. In addition, we compare the SDM system with the optimal beamforming case with Gaussian approximation in [3]. Note that the optimal case can not be applied to the SDMA since orthogonality of unitary vectors cannot be satisfied if different users are assigned to each antenna. As shown in the figure, the performance of the SDM comes within 3dB compared with the optimal case. Also the performance of the SDMA in Fig. 2 approaches the optimal case. To perform the optimal eigen beamforming, channel information for all propagation paths should be fed back to the BS. Thus the amount of feedback overhead increases as $N_t$, $N_r$, and $K$ grow. In comparison, for MIMO systems with ZF receivers, only scalar SNRs or acceptable data rates of each stream are sufficient for the feedback information and the performance gap decreases by combining with the multiuser diversity. Therefore, the MIMO systems employing the ZF receiver are better suited for systems where the bandwidth for feedback is limited as a reasonable performance can be achieved with reduced feedback overhead.

V. CONCLUSION

In this paper, we analyze the performance of multiuser MIMO systems with ZF receivers. We first derive the closed form expressions of capacity for MIMO systems with the ZF receiver in multiuser scenarios. As shown in simulation result section, the derived expressions accurately describe capacity for the multiuser MIMO system. Also, we compare the SDM and the SDMA by obtaining the capacity bounds and show that the performance gain decreases as the number of antenna increases.

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